EXISTENCE OF LIMIT CYCLES
FOR COUPLED VAN DER POL EQUATIONS

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Abstract. In this paper, we consider the existence of limit cycles of coupled van der Pol equations by using $S^1$-degree theory due to Dylawerski et al., see [3].

1. Introduction

In this paper, we consider the existence of limit cycles of coupled van der Pol equations of the form

\[
\begin{align*}
\ddot{u}_1 + \varepsilon_1 (u_1^2 - a_1)\dot{u}_1 + c_{11}u_1 + c_{12}u_2 + c_{13}u_3 + \ldots + c_{1n}u_n &= 0, \\
\ddot{u}_2 + \varepsilon_2 (u_2^2 - a_2)\dot{u}_2 + c_{21}u_1 + c_{22}u_2 + c_{23}u_3 + \ldots + c_{2n}u_n &= 0, \\
\vdots \\
\ddot{u}_n + \varepsilon_n (u_n^2 - a_n)\dot{u}_n + c_{n1}u_1 + c_{n2}u_2 + c_{n3}u_3 + \ldots + c_{nn}u_n &= 0,
\end{align*}
\]

where $n \geq 1, \varepsilon_i > 0$ for each $i = 1, \ldots, n$, and $c_{ij} \in \mathbb{R}$ for all $1 \leq i, j \leq n$.

The coupled van der Pol equation has been studied as a model of self-excited systems. This kind of systems appears in a wide variety of mechanical, electronical and biological systems. A limit cycle is a nontrivial periodic solution of the autonomous system above. The existence of a limit cycle of one dimensional van der Pol equation

\[
\ddot{u} + \varepsilon (u^2 - 1)\dot{u} + u = 0 \tag{1.1}
\]

is well known. The proof of the existence of the limit cycle is based on the Poincaré-Bendixson theorem(cf. [6], [15]). In contrast to one dimensional case, the existence of limit cycles for coupled van der Pol equations is not yet established except some restrictive cases(cf. [17]). In the present paper, we prove the existence of limit cycles for coupled van der Pol equations by using $S^1$-degree theory, see [3]. To avoid unnecessary complexity, we restrict ourselves to the case $n = 2$. That is we consider the problem

\[
\begin{align*}
\ddot{u}_1 + \varepsilon_1 (u_1^2 - 1)\dot{u}_1 + u_1 + c_2u_2 &= 0, \\
\ddot{u}_2 + \varepsilon_2 (u_2^2 - 1)\dot{u}_2 + c_1u_1 + u_2 &= 0,
\end{align*}
\]

(P)

Our argument below remains valid for the case that $n > 2$. We impose that following condition on $c_1$ and $c_2$ :

\[
c_1 \cdot c_2 \in (0, 1) \cup (1, +\infty) \tag{A}
\]

We can now state our main results:

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**Theorem 1.1.** For any \( \alpha > \frac{1}{\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}} \), there exist \( \varepsilon_1, \varepsilon_2 > 0 \) such that problem (P) has a nontrivial periodic solution \( u \in C^2(\mathbb{R}, \mathbb{R}^2) \) with period \( 2\pi \alpha \).

**Theorem 1.2.** There exists \( C > 0 \) such that for any \( (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \) such that \( \max \{ \varepsilon_1, \varepsilon_2 \} < C \), problem (P) possesses a nontrivial periodic solution \( u \in C^2(\mathbb{R}, \mathbb{R}^2) \) with period \( 2\pi \alpha \) for some \( \alpha > 1 \).

For the convenience of the reader we have included some references related to the equivariant degree methods.

The first degree theory for admissible \( S^1 \)--equivariant gradient maps, which is a rational number, is due to Dancer [1]. Degree theories for \( S^1 \)--equivariant maps have been defined by Ize and Vignoli in [9], [11], [12]. A definition of degree theory for equivariant orthogonal maps (symmetries of any compact abelian Lie group are admitted) is due to Ize and Vignoli [12]. Finally, degree theory for \( G \)--equivariant gradient maps, where \( G \) is any compact Lie group, is due to Gęba [4].

**2. Preliminaries**

We denote by \( \langle \cdot, \cdot \rangle_{>2} \) the scalar product of \( L^2([0,2\pi], \mathbb{R}^2) \). Define

\[
\mathbb{H}_{\text{per}} = \{ v : \mathbb{R} \to \mathbb{R}^2 : v \text{ is absolutely continuous, } \langle \dot{v}, \dot{v} \rangle_{>2} < \infty \text{ and } v(t) = v(t + 2\pi) \ \forall \ t \in \mathbb{R} \},
\]

and scalar products \( \langle \cdot, \cdot \rangle_{\mathbb{H}_{\text{per}}} : \mathbb{H}_{\text{per}} \times \mathbb{H}_{\text{per}} \to \mathbb{R} \) as follows

\[
\langle w, v \rangle_{\mathbb{H}_{\text{per}}} = \langle w, v \rangle_{>2} + \langle \dot{w}, \dot{v} \rangle_{>2}.
\]

Let \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} = \{ e^{i\theta} : \theta \in \mathbb{R} \} \) be the group of complex numbers of module 1 with an action given by multiplication of complex numbers. For any fixed \( m \in \mathbb{N} \) we denote by \( \mathbb{Z}_m \) a cyclic group of order \( m \) and define homomorphism \( \rho_m : S^1 \to GL(2, \mathbb{R}) \) as follows

\[
\rho_m(e^{i\theta}) = \begin{bmatrix} \cos(m\theta) & -\sin(m\theta) \\ \sin(m\theta) & \cos(m\theta) \end{bmatrix}.
\]

Of course \( \mathbb{R}[1, m] := (\mathbb{R}^2, \rho_m) \) is a two-dimensional representation of the group \( S^1 \). We will denote by \( \mathbb{R}[k, m] \) the direct sum of \( k \) copies of representation \( \mathbb{R}[1, m] \) and by \( \mathbb{R}[k, 0] \) \( k \)--dimensional trivial representation of the group \( S^1 \). Define action \( \rho : S^1 \times \mathbb{H}_{\text{per}} \to \mathbb{H}_{\text{per}} \) of the group \( S^1 \) as follows

\[
\rho(e^{i\theta}, v(t)) = v(t + \theta)
\]

(2.1)

In the following fact we collect some well known properties of the space \( \mathbb{H}_{\text{per}} \).

**Fact 2.1.** Under the above assumptions:

1. \( (\mathbb{H}_{\text{per}}, \langle \cdot, \cdot \rangle_{\mathbb{H}_{\text{per}}}) \) is a separable Hilbert space,
2. \( (\mathbb{H}_{\text{per}}, \langle \cdot, \cdot \rangle_{\mathbb{H}_{\text{per}}}) \) is an orthogonal representation of the group \( S^1 \) with \( S^1 \)--action given by (2.1),
Since $\mathbb{Z}_2 \subseteq S^1$, one can consider $\mathbb{H}_{\text{per}}$ as a $\mathbb{Z}_2$–space. Let $(\mathbb{H}_{\text{per}})^{\mathbb{Z}_2}$ denote the set of fixed points of the action of the group $\mathbb{Z}_2$ on $\mathbb{H}_{\text{per}}$. Moreover, by $(\mathbb{H}_{\text{per}})^{\mathbb{Z}_2}_\perp$ we denote the orthogonal complement of $(\mathbb{H}_{\text{per}})^{\mathbb{Z}_2}$ in $\mathbb{H}_{\text{per}}$.

Define

$$
\mathbb{H} = \{ v : \mathbb{R} \to \mathbb{R}^2 : v \text{ is absolutely continuous, } <\dot{v}, \dot{v}>_2 < \infty \text{ and } v(t) = -v(\pi + t) \ \forall \ t \in \mathbb{R} \}.
$$

In the following fact we collect some well known properties of the space $\mathbb{H}$.

**Fact 2.2.** Under the above assumptions:

1. $\mathbb{H} = \left( (\mathbb{H}_{\text{per}})^{\mathbb{Z}_2} \right)_\perp$,
2. $(\mathbb{H}, <\cdot, \cdot>)_{\mathbb{H}}$ is a separable Hilbert space,
3. $(\mathbb{H}, <\cdot, \cdot>)_{\mathbb{H}}$ is an orthogonal representation of the group $S^1$ with $S^1$–action given by the restriction of (2.1),
4. $\mathbb{H} = \text{cl} \left( \bigoplus_{n=1}^{\infty} \mathbb{R}[2,2n-1] \right)$.

Let $v = (v_1, v_2)$ be a periodic solution of (P) with period $2\pi \alpha$ for some $\alpha > 1$. Then by putting $t = \alpha \tau$ and $u(\tau) = (u_1(\tau), u_2(\tau)) = (v_1(\alpha \tau), v_2(\alpha \tau))$, we find that $u = (u_1, u_2) \in \mathbb{H}$ is a $2\pi$–periodic solution of problem

$$
\begin{aligned}
\dot{u}_1 + \varepsilon_1 \alpha (u_1^2 - 1) \dot{u}_1 + \alpha^2 (u_1 + c_2 u_2) &= 0, \\
\dot{u}_2 + \varepsilon_2 \alpha (u_2^2 - 1) \dot{u}_2 + \alpha^2 (c_1 u_1 + u_2) &= 0,
\end{aligned}
\tag{2.2}
$$

Here we put

$$
F(u) = \begin{pmatrix}
\varepsilon_1 \left( \frac{1}{3} u_1^3 - u_1 \right) \\
\varepsilon_2 \left( \frac{1}{3} u_2^3 - u_2 \right)
\end{pmatrix}, \quad A = \begin{pmatrix} 1 & c_2 \\ c_1 & 1 \end{pmatrix}.
$$

In this way we have converted problem (P) of finding of periodic solutions of any period into 1–parameter problem (2.2) of finding of periodic solutions of a fixed period $2\pi$.

**Lemma 2.1.** Let Hilbert space $\mathbb{H}$ and operator $F$ will be defined as above. Then,

1. $F : \mathbb{H} \to \mathbb{H}$ is well-defined continuous operator,
2. for any $w \in \mathbb{H}$ the following holds true:
   (a) $w(t) = w(t + 2\pi)$, for any $t \in \mathbb{R}$,
   (b) $\int_{0}^{2\pi} w(t) dt = 0$,
   (c) $\int_{0}^{2\pi} w(\tau) d\tau \in \mathbb{H}_{\text{per}},$
   (d) $< \int_{0}^{t} w(\tau) d\tau, (\cos(2nt), \cos(2nt)) >_{\mathbb{H}_{\text{per}}} = < \int_{0}^{t} w(\tau) d\tau, (\sin(2nt), \sin(2nt)) >_{\mathbb{H}_{\text{per}}} = 0$, for any $n \in \mathbb{N}$.
3. if $w \in \mathbb{H}$, then $\int_{0}^{t} F(w(\tau)) d\tau \in \mathbb{R}[2,0] \oplus \mathbb{H}$.

**Proof.** The easy proof is left to the reader. □
Notice that (2.2) can be rewritten as
\[ \ddot{u} + \alpha \frac{d}{dt} F(u) + \alpha^2 Au = 0. \] (2.3)

We will find a solution \( u = (u_1, u_2) \in \mathbb{H} \) of (2.2). We denote by \( L_1 : \mathbb{H} \to \mathbb{H} \) the inverse of the mapping \( u \to -\dot{u} \) for \( u \in \mathbb{H} \). That is \( u = L_1(-\dot{u}) \) for \( u \in \mathbb{H} \). We also put \( Lu = L_1Au \) for each \( u \in \mathbb{H} \). Denote by \( \sigma(L) \) the spectrum of \( L \). If \( \mu \in \sigma(L) \) then \( V(\mu) \) denotes the eigenspace of \( L \) corresponding to the eigenvalue \( \mu \). Notice that to describe eigenvalues and eigenspaces of the operator \( L \) it is enough to consider equation \( \ddot{u} + \frac{1}{\mu}Au = 0 \).

**Fact 2.3.** Suppose that \( c_1c_2 \in (0, 1) \cup (1, +\infty) \). Then, \( \sigma(L) = \bigcup_{n \in \mathbb{N}} \left\{ \mu_n^\pm = \frac{1 \pm \sqrt{c_1c_2}}{2n-1} \right\} \). Moreover,

1. \( V(\mu_n^+) \cong \mathbb{R}[1, 2n-1] \),
2. \( \mathbb{H} = cl \left( \bigoplus_{i=1}^{\infty} (V(\mu_n^-) \oplus V(\mu_n^+)) \right) \),
3. if \( c_1c_2 \in (0, 1) \), then \( \mu_1^+ > \mu_1^- > \mu_2^+ > \mu_2^- > \ldots > \mu_n^+ > \mu_n^- > \ldots > 0 \),
4. if \( c_1c_2 > 1 \), then \( \mu_1^+ > \mu_2^+ > \ldots > \mu_n^+ > \ldots > 0 \) and \( \mu_n^- < 0 \) for any \( n \in \mathbb{N} \).

Let \( m, M \in \mathbb{R} \) be such that \( 0 < m < M \). Let \( \eta : \mathbb{R} \to [0, 1] \) be a smooth function such that
\[ \eta(t) = \begin{cases} 0, & \text{if } \|t\| \leq \frac{m^2}{2}, \\ 1, & \text{if } \|t\| \geq \frac{M^2}{2}, \end{cases} \]
and that \( \eta'(t) > 0 \) for \( t \in \left( \frac{m^2}{2}, \frac{M^2}{2} \right) \). Moreover, define a smooth \( S^1 \)-equivariant function \( \theta : \mathbb{H} \to [0, 1] \) by the following formula
\[ \theta(u) = \eta \left( \frac{\|u\|^2}{2} \right) \] (2.4)

Denote by \( \pi : \mathbb{R}[2, 0] \oplus \mathbb{H} \to \mathbb{H} \) the \( S^1 \)-equivariant orthogonal projection. Since Lemma 2.1, for each \( \alpha > 0 \) and \( \delta \in [0, 1] \), we define a mapping \( G(\cdot, \alpha, \delta) : \mathbb{H} \to \mathbb{H} \) by
\[ G(v, \alpha, \delta) = -\delta \alpha \pi \left( \int_0^t F(v(\tau))d\tau \right) + \alpha^2 \theta(v)L(v) \] (2.5)

Then each solution \( u \in \mathbb{H} \) of problem \( G(u, \alpha, \delta) = u \) for some \( (\alpha, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+ \) satisfies
\[ \ddot{u} + \delta \alpha \frac{d}{dt} F(u) + \alpha^2 \theta(u)Au = 0 \] (2.6)

We will also consider the following family of differential equations
\[ \ddot{u} + \delta \alpha \frac{d}{dt} F(u) + \alpha^2 Au = 0 \] (2.7)

Below we formulate and prove three technical lemmas which we will apply in the next sections.

**Lemma 2.2.** There exists a monotone increasing function \( m(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for each \( \delta \in (0, 1] \), \( \alpha \in \mathbb{R}^+ \) and each solution \( u \in \mathbb{H} \) of problem (2.7) inequality \( \|u\| \leq m(\alpha) \) holds.
Proof. Fix $\alpha \in \mathbb{R}^+$ and $\delta \in (0, 1]$. Let $u \in \mathbb{H}$ be a solution of (2.7). Multiplying (2.7) by $\dot{u}$ and integrating over $[0, 2\pi]$, we find that

$$
\begin{cases}
\delta \alpha \varepsilon_1 \int_0^{2\pi} (u_1^2 - 1) \dot{u}_1^2 + \alpha^2 c_2 \int_0^{2\pi} \dot{u}_1 u_2 = 0, \\
\delta \alpha \varepsilon_2 \int_0^{2\pi} (u_2^2 - 1) \dot{u}_2^2 + \alpha^2 c_1 \int_0^{2\pi} u_1 \dot{u}_2 = 0.
\end{cases}
$$

Then we have that

$$
c_1 \varepsilon_1 \int_0^{2\pi} (u_1^2 - 1) \dot{u}_1^2 + c_2 \varepsilon_2 \int_0^{2\pi} (u_2^2 - 1) \dot{u}_2^2 = 0
$$

That is

$$
c_1 \varepsilon_1 \int_0^{2\pi} u_1^2 \dot{u}_1^2 + c_2 \varepsilon_2 \int_0^{2\pi} u_2^2 \dot{u}_2^2 = c_1 \varepsilon_1 \int_0^{2\pi} \dot{u}_1^2 + c_2 \varepsilon_2 \int_0^{2\pi} \dot{u}_2^2.
$$

Since $c_1 c_2 > 0$, it follows that there exists $C_1 > 0$ such that

$$
\int_0^{2\pi} u_1^2 \dot{u}_1^2 + u_2^2 \dot{u}_2^2 \leq C_1 \int_0^{2\pi} \dot{u}_1^2 + \dot{u}_2^2
$$

On the other hand by the Schwartz inequality, we have

$$
|u_1(t)|^2 \leq 2 \int_{s_1}^t |u_1 \dot{u}_1| \, dt \leq 2\sqrt{2\pi} \left( \int_0^{2\pi} |u_1|^2 |\dot{u}_1|^2 \right)^{1/2}
$$

and

$$
|u_2(t)|^2 \leq 2 \int_{s_2}^t |u_2 \dot{u}_2| \, dt \leq 2\sqrt{2\pi} \left( \int_0^{2\pi} |u_2|^2 |\dot{u}_2|^2 \right)^{1/2}
$$

for all $t \in [0, 2\pi]$, where $s_1, s_2 \in [0, 2\pi]$ satisfy $u_1(s_1) = u_2(s_2) = 0$. It then follows from (2.9) and the inequalities above that there exists $C_2 > 0$ such that

$$
|u(t)|^2 \leq C_2 |\dot{u}|_2 \quad \text{for all } t \in [0, 2\pi]
$$

We next multiply (2.7) by $u$ and integrate over $[0, 2\pi]$. Then we have that

$$
|\dot{u}|_2^2 = \alpha^2 \langle Au, u \rangle \leq \alpha^2 \|A\| |u|_2^2,
$$

where $\|A\|$ denotes the operator norm of matrix $A$. Then by (2.10), we have $|u|_2^2 \leq 2\pi C_2 |\dot{u}|_2 \leq 2\pi C_2 \alpha \sqrt{\|A\|} |u|_2$. It then follows from the inequality above that $|u|_2 \leq 2\pi C_2 \alpha \|A\|$ and then $|\dot{u}|_2 \leq 2\pi C_2 \alpha \sqrt{\|A\|}$. Then by putting $m(\alpha) = 2\pi C_2 \sqrt{\|A\|} \alpha (1 + \sqrt{\|A\|} \alpha)$ for each $\alpha \in \mathbb{R}^+$, we reach to the assertion.

Lemma 2.3. For each $\alpha > 0$, there exists $\delta_1(\alpha') > 0$ such that the problem (2.7) has no nontrivial solution in $\mathbb{H}$ for each $\alpha \in (0, \alpha')$ and $\delta > \delta_1(\alpha')$.

Proof. Fix $\delta > 0, \alpha' > 0$ and $\alpha \in (0, \alpha')$. Let $u \in \mathbb{H}$ be a solution of (2.7). Multiplying (2.7) by $\dot{u}$ and integrating over $[0, 2\pi]$ we find that

$$
c_1 \varepsilon_1 \int_0^{2\pi} (u_1^2 - 1) \dot{u}_1^2 + c_2 \varepsilon_2 \int_0^{2\pi} (u_2^2 - 1) \dot{u}_2^2 = 0.
$$

Since $u(t) = -u(t + \pi)$ and the above, there exists $t_0 \in [0, 2\pi]$ such that $u_1(t_0) = 1$ or $u_2(t_0) = 1$. Without loss of the generality one can assume that $u_1(t_0) = 1$ holds. We also may assume that $\dot{u}_1(t_0) \leq 0$. Suppose that $\dot{u}_1(t_0) > 0$. Since $\dot{u}_1(t_0) > 0$, $u_1(t_0) = 1$ and $u_1(t_0 + \pi) = -1$, there
exists \( t_0' > t_0 \) such that \( u_1(t_0') = 1 \) and \( \dot{u}_1(t_0') \leq 0 \). Then we assume that \( \dot{u}_1(t_0) \leq 0 \). We next integrate (2.7) over \([t_0 - \pi, t_0]\). Then notice that \( \dot{u}(t) = -\dot{u}(t - \pi) \) and \( u(t) = -u(t - \pi) \), we have

\[
2\dot{u}_1(t_0) + 2\alpha\delta\epsilon_1 \left( \frac{1}{3} u_1^3(t_0) - u_1(t_0) \right) = 2\dot{u}_1(t_0) - \frac{4}{3} \alpha\delta\epsilon_1 = -\alpha^2 \int_{t_0 - \pi}^{t_0} (u_1(t) + c_2 u_2(t)) dt.
\]

From the above and Lemma 2.2, by the Schwartz inequality we obtain the following

\[
0 \geq \dot{u}_1(t_0) = \frac{2}{3} \alpha\delta\epsilon_1 - \frac{\alpha^2}{2} \int_{t_0 - \pi}^{t_0} (u_1(t) + c_2 u_2(t)) dt \geq \alpha \left( \frac{2}{3} \delta\epsilon_1 - \alpha' \alpha \int_{t_0 - \pi}^{t_0} (|u_1(t)| + |c_2| |u_2(t)|) dt \right) \geq \alpha \left( \frac{2}{3} \delta\epsilon_1 - \frac{\alpha'}{2} \max\{1, |c_2|\} \int_{t_0 - \pi}^{t_0} (|u_1(t)| + |u_2(t)|) dt \right) \geq \alpha \left( \frac{2}{3} \delta\epsilon_1 - \frac{\alpha'}{2} \max\{1, |c_2|\} \sqrt{\pi} \|u\| \right) \geq \alpha \left( \frac{2}{3} \delta\epsilon_1 - \alpha' \max\{1, |c_2|\} \sqrt{\pi} m(\alpha') \right).
\]

Therefore to complete the proof it is enough to put

\[
\delta_1(\alpha') = \frac{3}{2\epsilon_1} \alpha' \max\{1, |c_2|\} \sqrt{\pi} m(\alpha').
\]

\( \square \)

**Lemma 2.4.** For each \( \frac{1}{\alpha^2} \in \mathbb{R}^+ \setminus \sigma(L) \), there exists \( \delta_2(\alpha) > 0 \) such that there exists no nontrivial solution of (2.7) in \( \mathbb{H} \) for all \( \delta \in (0, \delta_2(\alpha)) \).

**Proof.** Fix \( \frac{1}{\alpha^2} \in \mathbb{R}^+ \setminus \sigma(L) \). Suppose contrary to our claim that there exists a sequence \( \{(u_n, \delta_n)\} \subset \mathbb{H} \times \mathbb{R}^+ \) such that \( \lim_{n \to \infty} \delta_n = 0 \) and each \( u_n \) is a solution of (2.7) with \( \delta = \delta_n \).

Then by Lemma 2.2, sequence \( \{u_n\} \) is bounded in \( \mathbb{H} \). Therefore we may assume that \( u_n \) converges to \( u \in \mathbb{H} \) weakly in \( \mathbb{H} \) and strongly in \( L^2([0, 2\pi]; \mathbb{R}^2) \). Then since \( \sup_{t \in [0, 2\pi]} |u_n(t)| \geq 1 \) for each \( n \geq 1 \), we find that \( u \neq 0 \). Also one can see that \( u \) satisfies \( \ddot{u} + \alpha^2 Au = 0 \). That is \( Lu = \frac{1}{\alpha^2} u \).

Since \( \frac{1}{\alpha^2} \notin \sigma(L) \), this is a contradiction. Then the assertion holds. \( \square \)

### 3. \( S^1 \)-degree

Denote by \( \Gamma_0 \) the free abelian group generated by \( \mathbb{N} \) and let \( \Gamma = \mathbb{Z}_2 \oplus \Gamma_0 \). Then \( \gamma \in \Gamma \) means \( \gamma = \{\gamma_r\} \), where \( \gamma_0 \in \mathbb{Z}_2 \) and \( \gamma_r \in \mathbb{Z} \) for \( r \in \mathbb{N} \). Let \( V \) be a Hilbert space which is a representation of \( S^1 \). For each proper subgroup \( Q \subseteq \mathbb{H} \) and each \( S^1 \)-equivariant subset \( X \) of \( V \), we denote \( X^Q \) the subset of fixed points of \( Q \) in \( X \). For each \( U \subset V \subset \mathbb{R} \) and each \( S^1 \) equivariant compact mapping \( f : U \to V \), we define, by using the fact that there is a one-to-one correspondence between \( \mathbb{H} \) and the proper, closed subgroups \( Q \subseteq \mathbb{H} \), \( \text{Deg}(I - f, U) = \{\gamma_r\} \in \Gamma \) by \( \gamma_0 = \text{deg}_{S^1}(I - f, U) \) and \( \gamma_r = \text{deg}_{S^1}(I - f, U), r \in \mathbb{N} \) (cf. [3] and [16]). Next theorem has been formulated and proved in [3] and describe properties of \( S^1 \)-degree.
Lemma 3.5. Since (3.1) is bounded, invariant subset of $V$, then $\text{Deg}(I - f, U)$ satisfies (3.3).

Let $U$ be a Hilbert space which is a representation of $S^1$, $U$ be an open bounded, invariant subset of $V \oplus \mathbb{R}$ and $f : U \to V$ is a compact $S^1$-mapping such that $(I - f)(\partial U) \subset V \setminus \{0\}$. Then there exists a $\Gamma$-valued function $\text{Deg}(I - f, U)$ called $S^1$-degree, satisfying the following properties:

(a) if $\text{deg}_Q(I - f, U) \neq 0$, then $(I - f)^{-1}(0) \cap U^Q \neq \phi$,
(b) if $U_0 \subset U$ is open, invariant and $(I - f)^{-1}(0) \cap U \subset U_0$, then
\[
\text{Deg}(I - f, U) = \text{Deg}(I - f, U_0);
\]
(c) if $h : cl(U) \times [0, 1] \to V$ is an $S^1$-equivariant homotopy of compact mappings such that $(I - h)(\partial U \times [0, 1]) \subset V \setminus \{0\}$, Then
\[
\text{Deg}(I - h_0, U) = \text{Deg}(I - h_1, U).
\]

To apply $S^1$-degree theory to our problem, we need some definitions. Since Lemma 2.1, we define bounded operator $E : \mathbb{H} \to \mathbb{H}$ as follows
\[
E(v) = \pi \left( \begin{array}{c} \varepsilon_1 \int_0^t v_1 dt \\ \varepsilon_2 \int_0^t v_2 dt \end{array} \right) \text{ for each } v = (v_1, v_2) \in \mathbb{H}.
\]

For each $\alpha > 0$ and $\delta \in [0, 1]$, we define a mapping $H(\cdot, \cdot, \alpha, \delta) : \mathbb{H} \oplus \mathbb{R} \to \mathbb{H}$ by
\[
H(u, \lambda, \alpha, \delta) = G(u, \alpha, \delta) + \lambda \alpha E(u).
\]

It is easy to see that $H(\cdot, \cdot, \alpha, \delta)$ is an $S^1$-equivariant compact mapping. One can see that if $u \in \mathbb{H}$ satisfies $u = H(\cdot, \cdot, \alpha, \delta)$ for $(\alpha, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+$ then
\[
\ddot{u} + \delta \alpha \frac{d}{dt} F(u) + \alpha^2 \theta(u) Au = \lambda \alpha \frac{d^2}{dt^2} E(u)
\]
\[
(3.1)
\]
or in equivalent form
\[
\begin{array}{l}
\ddot{u}_1 + \delta \varepsilon_1 \alpha (u_2^2 - 1) \dot{u}_1 + \alpha^2 \theta(u)(u_1 + c_2 u_2) = \varepsilon_1 \alpha \lambda \dot{u}_1,
\ddot{u}_2 + \delta \varepsilon_2 \alpha (u_2^2 - 1) \dot{u}_2 + \alpha^2 \theta(u)(c_1 u_1 + u_2) = \varepsilon_2 \alpha \lambda \dot{u}_2,
\end{array}
\]
\[
(3.2)
\]

Lemma 3.5. Let $\alpha, \delta > 0$ and $\lambda \in \mathbb{R}$. Suppose that $u = (u_1, u_2) \in \mathbb{H}$ is a nontrivial solution of (3.1). Then $\lambda + \delta > 0$ and $w = \sqrt{\frac{\delta}{\lambda + \delta}} u$ is a solution of equation
\[
\dot{w} + (\lambda + \delta) \alpha \frac{d}{dt} F(w) + \alpha^2 \theta(u) Aw = 0
\]
\[
(3.3)
\]

Proof. Let $u \in \mathbb{H}$ be a nontrivial solution of (3.1). First of all we will show that $\lambda + \delta > 0$. Notice that (3.2) can be represented in the following form
\[
\begin{array}{l}
\ddot{u}_1 + \varepsilon_1 \alpha (\delta u_1^2 - (\lambda + \delta)) \dot{u}_1 + \alpha^2 \theta(u)(u_1 + c_2 u_2) = 0,
\ddot{u}_2 + \varepsilon_2 \alpha (\delta u_2^2 - (\lambda + \delta)) \dot{u}_2 + \alpha^2 \theta(u)(c_1 u_1 + u_2) = 0,
\end{array}
\]
\[
(3.4)
\]
Multiplying (3.4) by $(c_1 \dot{u}_1, c_2 \dot{u}_2)$ and integrating over $[0, 2\pi]$ we obtain the following
\[
c_1 \varepsilon_1 (u_1^2 - (\lambda + \delta)) u_1^2 + c_2 \varepsilon_2 (u_2^2 - (\lambda + \delta)) u_2^2 = 0.
\]

Since $u$ is a nonzero function, we obtain $\lambda + \delta > 0$. Putting $u = \sqrt{\frac{\lambda + \delta}{\delta}} w$ in (3.4) we find that $w$ satisfies (3.3). □
Lemma 3.6. Let $\alpha > 0$ be such that $\frac{1}{\alpha^2} \notin \sigma(L)$ and $n_0 = \max_{n \in \mathbb{N}} \{ \mu^+_n > \frac{1}{\alpha^2} \}$. Then,

(1) if $c_1c_2 \in (0, 1)$, then

\[
\text{deg}_Q(Id - H(\cdot, \cdot, \alpha, 0), U) = \begin{cases} 
0, & Q = S^1, \\
2, & Q = \mathbb{Z}_{2m-1} \text{ for } m \in \{1, \ldots, n_0 - 1\}, \\
2, & Q = \mathbb{Z}_{2n_0-1} \text{ and } \mu^-_{n_0} > \frac{1}{\alpha^2}, \\
1, & Q = \mathbb{Z}_{2n_0-1} \text{ and } \mu^-_{n_0} < \frac{1}{\alpha^2}, \\
0, & \text{otherwise,}
\end{cases}
\]

where $U = \{ u \in \mathbb{H} : m < \|u\| < M \} \times [-1, 1],$

(2) if $c_1c_2 > 1$, then

\[
\text{deg}_Q(Id - H(\cdot, \cdot, \alpha, 0), U) = \begin{cases} 
0, & Q = S^1, \\
1, & Q = \mathbb{Z}_{2m-1} \text{ for } m \in \{1, \ldots, n_0\}, \\
0, & \text{otherwise,}
\end{cases}
\]

where $U = \{ u \in \mathbb{H} : m < \|u\| < M \} \times [-1, 1].$

Proof. 1. Without loss of the generality one can assume that $\mu^-_{n_0} > \frac{1}{\alpha^2}$. The assertion is a slight modification of Corollary 4.7 of [16] and the proof is basically the same as that of Theorem 4.2 of [16]. Then we just show the sketch of the proof. From the definition $H(u, \lambda, \alpha, 0) = \alpha^2 \theta(u)Lu + \lambda \alpha E(u)$. Then one can see that $u = H(u, \lambda, \alpha, 0)$ if and only if $u = \alpha^2 \theta(u)Lu$. Moreover, since $\frac{1}{\alpha^2} \notin \sigma(L)$, $0 < \theta(u) < 1$ and therefore $u \in U$. It is easy to verify that the set $\{ u \in U : u = \alpha^2 \theta(u)Lu \}$ consists of a finite number of $S^1$ -orbits and is defined as follows

\[
\{ u \in U : u = \alpha^2 \theta(u)Lu \} = \bigcup_{i=1}^{n_0} \bigcup_{\nu \in \{-, +\}} \left\{ u \in V(\mu^\nu_i) : \mu^\nu_i \theta(u) = \frac{1}{\alpha^2} \right\}.
\]

Let $U^\pm_i \subset \text{cl}(U^\pm_i) \subset U, i = 1, \ldots, n_0$, be open, disjoint, $S^1$-invariant sets such that

\[
\left\{ u \in V(\mu^\nu_i) : \mu^\nu_i \theta(u) = \frac{1}{\alpha^2} \right\} \subset U^\pm_i, i = 1, \ldots, n_0.
\]

Therefore,

\[
\text{Deg}(Id - H(\cdot, \cdot, \alpha, 0), U) = \sum_{i=1}^{n_0} \sum_{\nu \in \{-, +\}} \text{Deg}(Id - H(\cdot, \cdot, \alpha, 0), U^\pm_i).
\]
Fix $i_0 \in \{1, \ldots, n_0\}$ and $\nu_0 \in \{-, +\}$. What is left is to show that

$$deg_Q(Id - H(\cdot, \cdot, \alpha, 0), U_{i_0}^{\nu_0}) = \begin{cases} 1, & \text{if } Q = \mathbb{Z}_{2i_0-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Fix $(v_0, 0, 0) \in U_{i_0}^{\nu_0} \times [-1, 1] \subset \mathbb{H} \times [-1, 1] = \left( V \left( \mu_{i_0}^{\nu_0} \right) \oplus \left( V \left( \mu_{i_0}^{\nu_0} \right) \right)^\perp \right) \times [-1, 1]$ such that $\alpha^2 \theta(v_0) = 1/\mu_{i_0}^{\nu_0}$. It is easy to show that $D(u-H(u, \lambda, \alpha, 0))(v_0, 0, 0)$ is a surjection. Finally applying Theorem 6.7 (a) of [3] we complete the proof.

2. The proof is literally the same as the proof of (1).

\[\square\]

## 4. Proof of Theorems

### Proof of Theorem 1.1.

Fix $\alpha > \frac{1}{\sqrt{1 + \sqrt{c_1c_2}}}$ and $M > m(\alpha)$. We also choose $m > 0$ so small that $\|u\|_{\infty} < 1$ for each $u \in \mathbb{H}$ with $\|u\| \leq m$. Choose $\lambda_0$ and $\delta_0$ such that $\lambda_0 - \delta_0 > \delta_1(\alpha)$ and,

$$[0, \lambda_0] \subset \{ \lambda : \lambda \geq \delta_1(\alpha) \} \cup \left\{ \lambda : 0 \leq \lambda \leq \frac{\delta_0 m^2}{m(\alpha)^2} \right\}$$

(4.1)

Let $U \subset \mathbb{H} \oplus \mathbb{R}$ be a open set defined by

$$U = \{ v \in \mathbb{H} : m < \|v\| < M \} \times (-\lambda_0, \lambda_0).$$

Then the boundary $\partial U$ of $U$ has the form $\partial U = B_1 \cup B_2 \cup B_3$, where

$$B_1 = \{ v \in \mathbb{H} : \|v\| = m \} \times [-\lambda_0, \lambda_0],$$

$$B_2 = \{ v \in \mathbb{H} : \|v\| = M \} \times [-\lambda_0, \lambda_0],$$

$$B_3 = \{ v \in \mathbb{H} : m < \|v\| < M \} \times (-\lambda_0, \lambda_0).$$

We put

$$S_1 = \{(u, \lambda) \in \text{cl}(U) : u = H(u, \lambda, \alpha, \delta) \text{ for some } \delta \in [0, \delta_0]\}.$$  

Then we claim that $S_1 \cap (B_1 \cup B_3) = \emptyset$. Suppose contrary to our claim that $\lambda = \pm \lambda_0$. Then $\theta(u) = 0$ by the definition. Then by (3.1), we have

$$\ddot{u} + \delta \alpha \frac{d}{dt} F(u) = \alpha \lambda \frac{d^2}{dt^2} E(u). \tag{4.2}$$

Multiplying (4.2) by $u$ and integrating over $[0, 2\pi]$, we find by the periodicity of $u$ that

$$\int_0^{2\pi} \dot{u}_1^2 = \int_0^{2\pi} \ddot{u}_1^2 = 0,$$

which implies $u \equiv 0$. This contradicts that $(u, \lambda) \in B_1 \subset (\mathbb{H} \setminus \{0\}) \times [-\lambda_0, \lambda_0]$. Suppose that $(u, \lambda) \in S_1 \cap B_3$. That is we assume that $\lambda = \pm \lambda_0$ and $m < \|u\| \leq M$. Fix $\lambda = -\lambda_0$. Since $\delta - \lambda_0 < 0$, from Lemma 3.5 it follows that $u = 0$, a contradiction. Suppose now that $\lambda = \lambda_0$. Since (3.2) holds with $\lambda = \lambda_0$, putting $\tilde{\delta} = \frac{\delta + \lambda_0}{\sqrt{\theta(u)}}$ and $w = \frac{\sqrt{\delta}}{\delta + \lambda_0} u$ we have that

$$\begin{cases} \ddot{w}_1 + \tilde{\delta} \alpha \tilde{\alpha} (w_1^2 - 1) \ddot{w}_1 + \tilde{\alpha}^2 (w_1 + c_2 w_2) = 0, \\ \ddot{w}_2 + \tilde{\delta} \alpha \tilde{\alpha} (w_2^2 - 1) \ddot{w}_2 + \tilde{\alpha}^2 (c_1 w_1 + w_2) = 0, \end{cases} \tag{4.3}$$

Therefore, we can conclude that $w \equiv 0$. This implies that $u \equiv 0$, contradicting the periodicity of $u$. Therefore, we have

$$\int_0^{2\pi} \ddot{w}_1 = \int_0^{2\pi} \ddot{w}_2 = 0,$$

which implies $w \equiv 0$. This contradicts that $(u, \lambda) \in B_3 \subset (\mathbb{H} \setminus \{0\}) \times [-\lambda_0, \lambda_0]$. Therefore, $\lambda$ is not $\lambda_0$, a contradiction. Hence, we conclude that $\lambda = \pm \lambda_0$.
Notice that $\tilde{\alpha} = \alpha\sqrt{\theta(u)} \leq \alpha$ and $\tilde{\delta} = \delta + \lambda_0 \sqrt{\theta(u)} \geq \delta_1(\alpha)$. Therefore putting in Lemma 2.3 $\alpha = \tilde{\alpha}, \delta = \tilde{\delta}, \alpha' = \alpha$ we obtain that (4.3) can not hold. Notice that we have just shown that $S_1 \cap (B_1 \cup B_3) = \emptyset$. Notice that if $(u, \lambda) \in S_1 \cap B_2$, then taking into account that $\theta(u) = 1$, we obtain

$$\dot{w} + (\delta + \lambda)\alpha \frac{d}{dt} F(w) + \alpha^2 Aw = 0.$$ 

That is we have a solution of (P) with period $2\pi\alpha$ and with $\varepsilon_1, \varepsilon_2$ replaced with $((\delta + \lambda)\varepsilon_1, (\delta + \lambda)\varepsilon_2)$. We next define a homotopy $I : \mathbb{H} \times [0, 1] \rightarrow \mathbb{H}$ of $S^1$--equivariant compact mappings by

$$I(u, \lambda, s) = -\pi \left( \delta_0 \alpha \int_0^t F_s(v)dt \right) + a^2\theta(v)Lv + \alpha\lambda E(u)$$

where

$$F_s(u) = \left( \varepsilon_1 \left( \frac{1}{3} u_1^3 - (1 - s)u_1 \right) \right) \left( \frac{1}{3} u_2^3 - (1 - s)u_2 \right)$$

for $(u, s) \in \mathbb{H} \times [0, 1]$. It is easy to verify that $I$ is a homotopy of $S^1$--equivariant compact mappings. We put

$$S_2 = \{(u, \lambda) \in U : I(u, \lambda, s) = u \text{ for some } s \in [0, 1]\}.$$ 

Then one can see that $(u, \lambda) \in S_2$ if and only if

$$\begin{cases} \dot{u}_1 + \delta_0 \varepsilon_1 \alpha (u_1^2 - 1 + s)\dot{u}_1 + a^2\theta(u)(u_1 + c_2u_2) = \varepsilon_1 \alpha \lambda \dot{u}_1, \\ \dot{u}_2 + \delta_0 \varepsilon_2 \alpha (u_2^2 - 1 + s)\dot{u}_2 + a^2\theta(u)(c_1u_1 + u_2) = \varepsilon_2 \alpha \lambda \dot{u}_2, \end{cases}$$

for some $s \in [0, 1]$. Repeating reasoning given above we can show that $S_2 \cap (B_1 \cup B_3) = \phi$. It also follows that if $(u, \lambda) \in B_2 \cap S_2$, then (P) has a solution with period $2\pi\alpha$. Therefore to complete the proof it is enough to show that $(S_1 \cup S_2) \cap B_2 \neq \emptyset$.

We next show that there is no element $(u, \lambda) \in cl(U)$ which satisfies that $I(u, \lambda, 1) = u$. Suppose contrary that there exists $(u, \lambda) \in cl(U)$ satisfying $I(u, \lambda, 1) = u$. Repeating reasoning given in the proof of Lemma 3.5 we show that $\lambda > 0$. Then by putting $w = \sqrt{\frac{\delta_0}{\lambda}} u$ we have that,

$$\begin{cases} \dot{w}_1 + \left( \frac{\lambda \alpha}{\tilde{\alpha}} \right) \varepsilon_1 \tilde{\alpha} (w_1^2 - 1)\dot{w}_1 + \tilde{\alpha}^2 (w_1 + c_2w_2) = 0, \\ \dot{w}_2 + \left( \frac{\lambda \alpha}{\tilde{\alpha}} \right) \varepsilon_2 \tilde{\alpha} (w_2^2 - 1)\dot{w}_2 + \tilde{\alpha}^2 (c_1w_1 + w_2) = 0, \end{cases}$$

where $\tilde{\alpha} = \alpha\sqrt{\theta(u)}$. If $\lambda > \delta_1(\alpha)$, then $\delta = \frac{\lambda \alpha}{\tilde{\alpha}} = \frac{\lambda}{\sqrt{\theta(u)}} > \delta_1(\alpha)$. Since $\tilde{\alpha} = \alpha\sqrt{\theta(u)} < \alpha$, from Lemma 2.3 it follows that (4.5) can not hold. On the other hand, if $\delta_0m^2 > \lambda m(\alpha)^2$, then $\|w\| \geq \|u\|\sqrt{\frac{\delta_0}{\lambda}} > m(\alpha)$. Then by Lemma 2.2, we have that (4.5) can not hold. Thus we find that $I(u, \lambda, 1) \neq u$ for all $(u, \lambda) \in cl(U)$. We now define a homotopy of mapping $\Psi : U \times [0, 1] \rightarrow \mathbb{H}$ by

$$\Psi(u, \lambda, s) = \begin{cases} H(u, \lambda, \alpha, 2\delta_0) \text{ for } (u, \lambda, s) \in U \times [0, \frac{1}{2}], \\ I(u, \lambda, 2s - 1) \text{ for } (u, \lambda, s) \in U \times [\frac{1}{2}, 1]. \end{cases}$$
Then since $Ψ(u, λ, 0) = H(u, λ, \alpha, 0) = Lu + \alpha \lambda E(u)$. Since $α > \frac{1}{\sqrt{1 + c_1 c_2}}$, $μ_1^+ > \frac{1}{α^2}$ and Lemma 3.6, we obtain $Deg(I - Ψ(\cdot, 0), U) \neq 0$. Now suppose that $u \neq Ψ(u, λ, s)$ for all $u \in \partial U$ and $s \in [0, 1]$. Then by the homotopy invariance of $S^1$–degree we find that $Deg(I - Ψ(\cdot, 1), U) \neq 0$. Then by property (1) of Theorem 3, we have that there exists $(u, λ) \in U$ satisfying $u = Ψ(u, λ, 1)$. This contradicts to the observation above. Therefore we have that there exists $(u, λ) \in \partial U$ such that $u = Ψ(u, λ, s)$ for some $s \in [0, 1)$. On the other hand, we have, from the argument above, that there is no solution $(u, λ)$ of problem $u = Ψ(u, λ, s)$ on $B_1 \cup B_3$ for all $s \in [0, 1)$. Thus we obtain that there exists a solution $(u, λ)$ of problem $u = Ψ(u, λ, s)$ on $B_2$ for some $s \in [0, 1)$. Then by the argument above, we obtain a solution of (P) with period $2πα$.

**Proof of Theorem 1.2.** We choose $α_0^2 \in \left(\frac{1}{μ_1^+}, \frac{1}{μ}\right)$, where $μ = μ_1^+$ when $c_1 c_2 < 1$ and $μ = μ_2^+$ when $c_1 c_2 > 1$. Then by Lemma 2.4 there exists $δ_2(α_0) > 0$ such that the problem (2.7) with $α = α_0$ has no solution for $δ \in (0, δ_2(α_0)]$. We will show that for any $δ \in (0, δ_2(α_0))$, the problem (2.7) has a solution for some $α \in (0, α_0)$. Suppose contrary that there exists $δ_1 \in (0, δ_2(α_0))$ such that problem (2.7) has no solution with $δ = δ_1$ for any $α \in (0, α_0)$. Then we claim that there exists $ρ \in (0, δ_1)$ such that $δ_1 + ρ < δ_2(α_0)$ and for any $δ \in [δ_1 - ρ, δ_1 + ρ]$, problem (2.7) has no solution for any $α \in (0, α_0)$. Suppose that there exists a sequence $\{u_n, δ_n, α_n\} \in H \times R^+ \times (0, α_0)$ such that $\lim_{n→∞} δ_n = δ_1$ and each $u_n$ is a solution of problem (2.7) with $δ = δ_n$ and $α = α_n$. By Lemma 2.2 sequence $\{u_n\}$ is bounded in $H$. Therefore there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup u \in H$ weakly as $i → ∞$ and $\lim_{i→∞} α_{n_i} = α \in (0, α]$. One can see from (2.8) with $u = (u_1, u_2)$ replaced by $u_n = (u_{1n}, u_{2n})$ that $\|u_{1n}(t)\| \geq 1$ or $\|u_{2n}(t)\| \geq 1$ for some $t \in [0, 2π]$. That is $u$ is a nontrivial solution of (2.7) with $δ = δ_1$, a contradiction.

Choose $M > 0$ such that $M \sqrt{\frac{ρ}{ρ + δ_1}} > m(α_0)$. Put

$$U = \{v \in H : m < \|v\| < M\} \times (−δ_1, δ_1)$$

$$B_1 = \{v \in H : \|v\| = m\} \times [−δ_1, δ_1],$$

$$B_2 = \{v \in H : \|v\| = M\} \times [−δ_1, δ_1],$$

$$B_3 = \{v \in H : m < \|v\| < M\} \times [−δ_1, δ_1].$$

We first see that there is no nontrivial solution $u$ of problem $u = H(u, λ, α_0, (1 − s)ρ)$ for any $s \in [0, 1]$. That is there is no nontrivial solution of

$$\ddot{u} + (1 − s)ρα_0 \frac{d}{dt} F(u) + α_0^2 Au = λα_0 \frac{d^2}{dt^2} E(u) \quad (4.6)$$

Suppose that $u \in U$ is a solution of problem (4.6). If $s = 1$, then $u$ is a solution of

$$\ddot{u} − λα_0 \frac{d^2}{dt^2} E(u) + α_0^2 Au = 0 \quad (4.7)$$

Then multiplying (4.7) by $\dot{u}$ and integrating over $[0, 2π]$, we obtain $u = 0$, a contradiction. Suppose now that $s < 1$. In case $(1 − s)ρ + λ \leq 0$, problem (4.6) has no nontrivial solution by
Lemma 3.5. If \((1 - s)\rho + \lambda > 0\), then \(w = \sqrt{\frac{(1 - s)\rho}{(1 - s)\rho + \lambda}} u\) is a solution of
\[
\dot{w} + \delta \alpha_0 \frac{d}{dt} F(w) + \alpha^2 w = 0,
\]
where \(\delta = \lambda + (1 - s)\rho\). Since \(\delta \leq \delta_1 + \rho < \delta_2(\alpha_0)\), this contradicts to the definition of \(\delta_2(\alpha_0)\). Thus we find that (4.6) has no solution in \(U\). Define a homotopy of compact mappings \(\tilde{H} : U \times [0, 1] \to \mathbb{H}\) by
\[
\tilde{H}(u, \lambda, s) = H(u, \lambda, \alpha_0((1 - s) + s\theta(u)), (1 - s)\rho), \quad \text{for } (u, \lambda) \in U \quad \text{and } s \in [0, 1],
\]
where \(\theta : \mathbb{H} \to [0, 1]\) is given by (2.4). One can see that \(\tilde{H}(u, \lambda, s) = H(u, \lambda, \alpha_0, \rho)\) for \((u, \lambda) \in U\). We will see that there exists no solution \(u = \tilde{H}(u, \lambda, s)\) on \(\partial U\) for any \(s \in [0, 1]\). Suppose, contrary to our claim that \(u = \tilde{H}(u, \lambda, s)\) for some \(u \in \partial U\) and \(s \in [0, 1]\). Suppose that \(u \in B_2\). Then putting \(\alpha = \alpha_0((1 - s) + s\theta(u))\) and \(w = \sqrt{\frac{(1 - s)\rho}{(1 - s)\rho + \lambda}} u\) we have that \(w\) is a solution of
\[
\dot{w} + \delta \alpha \frac{d}{dt} F(w) + \alpha^2 w = 0 \quad (4.8)
\]
where \(\delta = (1 - s)\rho + \lambda\). Since \(\|u\| = M\), we have that
\[
\|w\| = \sqrt{\frac{(1 - s)\rho}{(1 - s)\rho + \lambda}}\|u\| = \sqrt{\frac{(1 - s)\rho}{(1 - s)\rho + \lambda}} > m(\alpha_0).
\]
On the other hand, we have by Lemma 2.2 that \(\|w\| < m(\alpha) \leq m(\alpha_0)\), a contradiction. Finally suppose that \(u \in B_3\). If \(\lambda = -\delta_1\), then we reach a contradiction by Lemma 3.5. Suppose that \(\lambda = \delta_1\). Then \(\delta = (1 - s)\rho + \delta_1 \in [\delta_1 - \rho, \delta_1 + \rho]\). Therefore by the assumption, (4.8) has no nontrivial solution. Summing up, we have shown that \(u \notin \partial U\). Therefore by the homotopy invariance of degree for \(S^1\)-equivariant maps we obtain that \(\text{Deg}(I - \tilde{H}(\cdot, \cdot, 0), U) = \text{Deg}(I - H(\cdot, \cdot, 1), U)\). Since \(I - \tilde{H}(\cdot, \cdot, 0) \neq 0\) on \(U\) by the same argument we have that \(\text{Deg}(I - \tilde{H}(\cdot, \cdot, 0), U) = 0\). On the other hand, noting that \(\tilde{H}(\cdot, \cdot, 1) = H(\cdot, \cdot, \alpha_0, 0)\), by Lemma 3.6 we obtain that \(\text{Deg}(I - \tilde{H}(\cdot, \cdot, 1), U) \neq 0\), a contradiction. Which completes the proof.

**References**


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