SOLUTIONS OF MULTIPARAMETER SYSTEMS OF ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract. In this article we study bifurcations of weak solutions of the following multiparameter variational system of elliptic differential equations:

\[
\begin{align*}
-\Delta u &= \Lambda u + \nabla_u \eta(u, \Lambda) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

We formulate necessary and sufficient conditions for the existence of bifurcation points and global bifurcation points of weak solutions of system (P).

1. INTRODUCTION

Nonlinear elliptic eigenvalue problem of the form

\[
\begin{align*}
-\Delta u &= \lambda u + \eta(u, \lambda) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is an open, bounded subset with boundary of the class \( C^{1-} \) and \( \eta(x, \lambda) = o(|x|) \) locally uniformly in \( \lambda \in \mathbb{R} \), has been extensively studied by many authors from the bifurcation theory point of view. One of the most famous applications of the Leray-Schauder degree is the Rabinowitz alternative, see [23]. This theorem has been successfully applied to the study of global bifurcations of solutions of (1.1). Let \( \sigma(-\Delta; \Omega) \) denote the set of eigenvalues of the Laplace operator \(-\Delta\) with respect to the Dirichlet boundary condition. Denote by \( V_{-\Delta}(\lambda_k) \) the eigenspace of the Laplace operator \(-\Delta\) corresponding to the eigenvalue \( \lambda_k \in \sigma(-\Delta; \Omega) \). It was proved that any characteristic eigenvalue \( \lambda_k^{-1} \) of the Laplace operator \(-\Delta\), such that \( \dim V_{-\Delta}(\lambda_k) \) is odd, is a global bifurcation point, in the sense of Rabinowitz, of weak solutions of (1.1). Since problem (1.1) possesses a gradient structure the Conley index theory, the Morse theory and the Lusternik-Schnirelman theory can be applied to the study of local bifurcations of solutions of (1.1), see for instance [2, 17, 24]. Namely, it was proved that any characteristic eigenvalue \( \lambda_k^{-1} \) of the Laplace operator \(-\Delta\) is a bifurcation point of weak solutions of (1.1). If we additionally assume that \( \Omega \subset \mathbb{R}^n \) is SO(2)-symmetric then any characteristic value \( \lambda_k^{-1} \) of the Laplace operator \(-\Delta\), such that \( V_{-\Delta}(\lambda_k) \) is a nontrivial SO(2)-representation, is a global bifurcation point.

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of weak solutions of (1.1), see [25, 27]. We emphasize that in SO(2)-symmetric case it can happen that characteristic eigenvalue \( \lambda_k \), such that \( \dim V_{-\Delta}(\lambda_k) \) is even, can be a global bifurcation point of weak solutions of (1.1). The degree for SO(2)-equivariant gradient maps has been applied to the study of solutions of elliptic differential equations in [10, 13, 25, 26, 28]. See also [15, 30] for definition and applications of equivariant versions of the Conley index theory and the Morse theory, respectively.

A multiparameter generalizations of problem (1.1) has been studied for instance in [4]-[8], [12, 14, 16, 31].

In this article we study a multiparameter variational generalization of (1.1). Namely, we consider the following multiparameter system of elliptic differential equations

\[
\begin{align*}
-\Delta u &= \Lambda u + \nabla u \eta(u, \Lambda) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (1.2)

where

(1) \( \Omega \subset \mathbb{R}^n \) is an open, bounded subset with \( C^1 \)-boundary,
(2) \( u = (u_1, \ldots, u_m) \),
(3) \( \Lambda \in S(m; \mathbb{R}) \), where \( S(m; \mathbb{R}) \) denotes the set of real, symmetric \( (m \times m) \)-matrices,
(4) \( \nabla u \eta(x, \Lambda) = o(|x|) \) locally uniformly in \( \Lambda \in S(m; \mathbb{R}) \),

and linearization of (1.2) at \((0, \Lambda)\) i.e.

\[
\begin{align*}
-\Delta u &= \Lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (1.3)

The matrix \( \Lambda \in S(m; \mathbb{R}) \) in systems (1.2), (1.3) is considered as a parameter. Our purpose is to study the set of bifurcation points, branching points and global bifurcation points of weak solutions of system (1.2). Suppose that \( \Omega \subset \mathbb{R}^n \) is SO(2)-symmetric. Since system (1.2) possesses variational structure, to study the weak solutions of (1.2) we apply the degree for SO(2)-equivariant gradient maps, see [25, 27]. In non-symmetric case we apply the Conley index technique, see [7], and the Leray-Schauder degree, see [18].

After introduction this article is organized as follows.

In Section 2 we have summarized without proofs the relevant material on the degree for SO(2)-equivariant gradient maps, defined in [25, 27], thus making our exposition self-contained.

In Section 3 we have discussed linear system (1.3) which is a linearization of (1.2) at \((0, \Lambda)\) \( \in \bigoplus_{i=1}^m H^1_0(\Omega) \times S(m, \mathbb{R}) \). In Subsection 3.1 we have discussed the generalized eigenspaces and eigenvalues of (1.3). In Proposition 3.1.1 we have collected the basic properties of these eigenspaces and eigenvalues. Since \( \dim S(2; \mathbb{R}) = 3 \), generalized eigenvalues have very nice geometrical interpretation. Hence in Subsection 3.2 we have discussed separately case \( m = 2 \). In Subsections 3.3, 3.4 we have defined a bifurcation index in terms of the degree for SO(2)-equivariant gradient maps, see formula (3.3.1). Moreover, we have proved sufficient conditions for the nontriviality of this index, see Lemmas 3.3.3, 3.4.3, Remark 3.3.1 and Corollary 3.4.1.
Section 4 is devoted to the study of bifurcations of weak solutions of system (1.2). This section contains the main results of our article. In Definition 4.1 we have introduced a notion of a bifurcation point, a branching point and a global bifurcation point of weak solutions of system (1.2).

In Subsection 4.1 have completely described the set of bifurcation points of weak solutions of system (1.2), see Theorem 4.1.1. Namely, we have proved that for any matrix \( \Lambda \in S(m; \mathbb{R}) \) satisfying \( \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \neq \emptyset \) a point \( (0, \Lambda) \in \bigoplus_{i=1}^{m} H^1_0(\Omega) \times S(m, \mathbb{R}) \) is a bifurcation points of weak solutions of system (1.2), where \( \sigma(\Lambda) \) denotes the spectrum of \( \Lambda \). Since the notion of a bifurcation point is local, first we have reduced the problem of studying of bifurcation points to a finite-dimensional one, see Lemma 4.1.1, and next, using the Conley index, have proved Theorem 4.1.1. In Theorem 4.1.2 we have proved that the set of global bifurcation points of weak solutions of system (1.2) is nonempty. In fact we have proved that any matrix \( \Lambda \in S(m; \mathbb{R}) \), such that there is \( \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \) satisfying \( \dim V_{\Lambda}(\lambda_k) \) is odd, is a global bifurcation point of weak solutions of system (1.2), where \( V_{\Lambda}(\lambda_k) \) is the eigenspace of \( \Lambda \) corresponding to the eigenvalue \( \lambda_k \). Finally, in Theorem 4.1.3 we have additionally assumed that \( \Omega \subset \mathbb{R}^n \) is SO(2)-invariant. We have proved that for any matrix \( \Lambda \in S(m; \mathbb{R}) \), such that there is \( \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \) satisfying alternative \( \dim V_{\Lambda}(\lambda_k) \) is odd or \( V_{\Lambda}(\lambda_k) \) is a nontrivial SO(2)-representation, is a global bifurcation point of weak solutions of system (1.2). To prove this theorem we have used the degree for SO(2)-equivariant gradient maps. The choice of the degree for SO(2)-equivariant gradient maps seems to be the best adapted to our theory. The advantage of using the degree for SO(2)-equivariant gradient maps lies in the fact that there is a matrix \( \Lambda \in S(m, \mathbb{R}) \) such that there is \( \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \) satisfying \( \dim V_{\Lambda}(\lambda_k) \) is even and \( V_{-\Delta}(\lambda_k) \) is a nontrivial SO(2)-representation. In this situation (see for instance Example 5.2) then Theorem 4.1.3 is stronger than Theorem 4.1.2.

In Subsection 4.2 we have considered case \( m = 2 \). In this case the set of bifurcation points and global bifurcation points are represented as a sum of cones numbered by the elements of \( \sigma(-\Delta; \Omega) \) i.e. by the eigenvalues of the Laplace operator \(-\Delta\).

In Section 5 we have illustrated the abstract results proved in this article.

2. Preliminaria

In this article we apply the Conley index, the Leray-Schauder topological degree and the degree for SO(2)-equivariant gradient maps. Since the first two topological invariants are well-known, in this section, to make this paper self-contained, we remind only the main properties of the degree for SO(2)-equivariant gradient maps.

Put \( U(SO(2)) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z} \) and define actions \( +, \star : U(SO(2)) \times U(SO(2)) \to U(SO(2)) \) as follows
\[ \alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k, \ldots) \] 

(2.1)

\[ \alpha \star \beta = (\alpha_0 \cdot \beta_0, \alpha_0 \cdot \beta_1 + \beta_0 \cdot \alpha_1, \ldots, \alpha_0 \cdot \beta_k + \beta_0 \cdot \alpha_k, \ldots) \] 

(2.2)

where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots), \beta = (\beta_0, \beta_1, \ldots, \beta_k, \ldots) \in U(SO(2)) \). It is easy to check that \((U(SO(2)), +, \star)\) is a commutative ring with unit \( \mathbb{I} = (1, 0, 0, \ldots) \in U(SO(2)) \). The ring \((U(SO(2)), +, \star)\) is known as the tom Dieck ring of the group \( SO(2) \). For a definition of the tom Dieck ring \( U(G) \) for any compact Lie group \( G \) we refer the reader to [11].

If \( \delta_1, \ldots, \delta_q \in U(SO(2)) \), then we write \( \prod_{j=1}^{q} \delta_j \) for \( \delta_1 \ast \cdots \ast \delta_q \). Moreover, it is understood that \( \prod_{j \in \emptyset} \delta_j = \mathbb{I} = (1, 0, \ldots) \in U(SO(2)) \).

Let \( V \) be a real, finite-dimensional, orthogonal \( SO(2) \)-representation and \( k \in \mathbb{N} \). Then we set

\[ C_{SO(2)}(V, \mathbb{R}) = \{ f \in C^k(V, \mathbb{R}) : f \text{ is } SO(2)\text{-invariant} \}, \]

\[ C_{SO(2)}^{-1}(V, V) = \{ f \in C^k(V, V) : f \text{ is } SO(2)\text{-equivariant} \}. \]

Let \( f \in C_{SO(2)}^1(V, \mathbb{R}) \). Since \( V \) is an orthogonal representation, \( \nabla f \in C_{SO(2)}^0(V, V) \). Choose an open, bounded and \( SO(2)\)-invariant subset \( \Omega \subset V \) and closed subgroup \( H \subset SO(2) \) and define \( \Omega^H = \{ v \in \Omega : g v = v \forall g \in H \} \). Assume additionally that \( (\nabla f)^{-1}(0) \cap \partial \Omega = \emptyset \).

Under these assumptions we have defined in [25] the degree for \( SO(2)\)-equivariant gradient maps \( \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) \in U(SO(2)) \) with coordinates

\[ \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) = \]

\[ = (\nabla_{SO(2)} - \text{deg}_{SO(2)}(\nabla f, \Omega), \nabla_{SO(2)} - \text{deg}_{Z_1}(\nabla f, \Omega), \ldots, \nabla_{SO(2)} - \text{deg}_{Z_k}(\nabla f, \Omega), \ldots). \]

Throughout this article \( \alpha > 0 \) and \( D_\alpha(V) = \{ v \in V : |v| < \alpha \} \). In the following theorem we formulate the main properties of the degree for \( SO(2)\)-equivariant gradient maps.

**Theorem 2.1** ([25]). Under the above assumptions the degree for \( SO(2)\)-equivariant gradient maps has the following properties:

1. if \( \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) \neq \emptyset \), then \( (\nabla f)^{-1}(0) \cap \Omega \neq \emptyset \).
2. if \( \nabla_{SO(2)} - \text{deg}_H(\nabla f, \Omega) \neq \emptyset \), then \( (\nabla f)^{-1}(0) \cap \Omega^H \neq \emptyset \).
3. if \( \Omega = \Omega_0 \cup \Omega_1 \) and \( \Omega_0 \cap \Omega_1 = \emptyset \), then
   \[ \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) = \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega_0) + \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega_1). \]
4. if \( \Omega_0 \subset \Omega \) is an open \( SO(2)\)-equivariant subset and \( (\nabla f)^{-1}(0) \cap \Omega \subset \Omega_0 \), then
   \[ \text{DEG}(\nabla f, \Omega) = \text{DEG}(\nabla f, \Omega_0). \]
5. if \( f \in C_{SO(2)}^1(V \times [0, 1], \mathbb{R}) \) is such that \( (\nabla_v f)^{-1}(0) \cap (\partial \Omega \times [0, 1]) = \emptyset \), then
   \[ \nabla_{SO(2)} - \text{deg}(\nabla_v f(\cdot, 0), \Omega) = \nabla_{SO(2)} - \text{deg}(\nabla_v f(\cdot, 1), \Omega), \]
6. if \( W \) is an orthogonal \( SO(2)\)-representation, then
   \[ \nabla_{SO(2)} - \text{deg}((\nabla f, Id), \Omega \times D_\alpha(W)) = \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega). \]
(7) if \( f \in C^2_{SO(2)}(V, \mathbb{R}) \) is such that \( \nabla f(0) = 0 \) and \( \nabla^2 f(0) \) is an \( SO(2) \)-equivariant self-adjoint isomorphism, then there is \( \alpha > 0 \) such that
\[
\nabla_{SO(2)} \deg(\nabla f, D_\alpha(V)) = \nabla_{SO(2)} \deg(\nabla^2 f(0), D_\alpha(V)).
\]

Below we formulate product formula for the degree for \( SO(2) \)-equivariant gradient maps.

**Theorem 2.2** ([27]). For \( i = 1, 2 \) assume that

1. \( \Omega_i \subset V_i \) is an open, bounded and \( SO(2) \)-invariant subset of \( SO(2) \)-representation \( V_i \),
2. \( f_i \in C^1_{SO(2)}(V_i, \mathbb{R}) \) satisfies condition \( (\nabla f_i)^{-1}(0) \cap \partial \Omega_i = \emptyset \).

Then,
\[
\nabla_{SO(2)} \deg((\nabla f_1, \nabla f_2), \Omega_1 \times \Omega_2) = \nabla_{SO(2)} \deg(\nabla f_1, \Omega_1) \circ \nabla_{SO(2)} \deg(\nabla f_2, \Omega_2).
\]

For \( j \in \mathbb{N} \) define a map \( \rho^j : SO(2) \to GL(2, \mathbb{R}) \) as follows
\[
\rho^j(e^{i\theta}) = \begin{bmatrix} \cos j \cdot \theta & -\sin j \cdot \theta \\ \sin j \cdot \theta & \cos j \cdot \theta \end{bmatrix}, \quad 0 \leq \theta < 2 \cdot \pi.
\]

For \( k, j \in \mathbb{N} \) we denote by \( \mathbb{R}[k, j] \) the direct sum of \( k \) copies of \( (\mathbb{R}^2, \rho^j) \), we also denote by \( \mathbb{R}[k, 0] \) the trivial \( k \)-dimensional \( SO(2) \)-representation. We say that two \( SO(2) \)-representations \( V \) and \( W \) are equivalent if there exists an\( SO(2) \)-equivariant, linear isomorphism \( T : V \to W \). The following classic result gives a complete classification (up to equivalence) of finite-dimensional \( SO(2) \)-representations (see [1]).

**Theorem 2.3** ([1]). If \( V \) is a finite-dimensional \( SO(2) \)-representation then there exist finite sequences \( \{k_i\}, \{j_i\} \) satisfying
\[
(*) \quad j_i \in \{0\} \cup \mathbb{N}, \quad k_i \in \mathbb{N}, \quad 1 \leq i \leq r, \ j_1 < j_2 < \cdots < j_r,
\]
such that \( V \) is equivalent to \( \bigoplus_{i=1}^r \mathbb{R}[k_i, j_i] \). Moreover, the equivalence class of \( V \) \( (V \approx \bigoplus_{i=1}^r \mathbb{R}[k_i, j_i]) \) is uniquely determined by \( \{j_i\}, \{k_i\} \) satisfying (*).

We will denote by \( m^{-}(L) \) the Morse index of a symmetric matrix \( L \). To apply successfully any topological degree we need computational formulas for this invariant. Below we show how to compute the degree for \( SO(2) \)-equivariant gradient maps of linear, self-adjoint, \( SO(2) \)-equivariant isomorphism.

**Lemma 2.1** ([25]). If \( V \approx \mathbb{R}[k_0, 0] \oplus \mathbb{R}[k_1, m_1] \oplus \cdots \oplus \mathbb{R}[k_r, m_r], \ L : V \to V \) is a self-adjoint, \( SO(2) \)-equivariant, linear isomorphism and \( \alpha > 0 \) then

1. \( L = \text{diag} (L_0, L_1, \ldots, L_r), \)
2. \[
\nabla_{SO(2)} \deg_H(L, D_\alpha(V)) = \begin{cases} (-1)^{m^{-}(L_0)}, & \text{for } H = SO(2), \\ (-1)^{m^{-}(L_0)} \cdot \frac{m^{-}(L_1)}{2}, & \text{for } H = \mathbb{Z}_{m_1}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_r}\}.
\end{cases}
\]
(3) in particular, if \( L = -Id \), then
\[
\nabla_{SO(2)} - \deg_H(-Id, D_\alpha(V)) = \begin{cases} 
(-1)^{k_0}, & \text{for } H = SO(2), \\
(-1)^{k_0} \cdot k_i, & \text{for } H = \mathbb{Z}_{m_i}, \\
0, & \text{for } H \not\in \{SO(2), \mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_r}\}. 
\end{cases}
\] (2.3)

Let \((\mathbb{H}, \langle \cdot, \cdot \rangle_\mathbb{H})\) be an infinite-dimensional, separable Hilbert space which is an orthogonal SO(2)-representation and let \( C^1_{SO(2)}(\mathbb{H}, \mathbb{R}) \) denote the set of SO(2)-invariant \( C^1 \)-functionals. Fix \( \Phi \in C^1_{SO(2)}(\mathbb{H}, \mathbb{R}) \) such that
\[
\nabla \Phi(u) = u - \nabla \eta(u),
\] (2.4)
where \( \nabla \eta : \mathbb{H} \to \mathbb{H} \) is an SO(2)-equivariant compact operator. Let \( \mathcal{U} \subset \mathbb{H} \) be an open bounded and SO(2)-invariant set such that \( (\nabla \Phi)^{-1}(0) \cap \partial \mathcal{U} = \emptyset \). In this situation \( \nabla_{SO(2)} - \deg(Id - \nabla \eta, \mathcal{U}) \in U(SO(2)) \) is well-defined, see [25].

Let \( L : \mathbb{H} \to \mathbb{H} \) be a linear, compact, bounded, self-adjoint, SO(2)-equivariant operator with spectrum \( \sigma(L) = \{ \lambda_i \}_{i=1}^{\infty} \). By \( V_L(\lambda_i) \) we will denote the eigenspace of \( L \) corresponding to the eigenvalue \( \lambda_i \) and by \( \mu_L(\lambda_i) \) the multiplicity of \( \lambda_i \). Since operator \( L \) is linear, compact, bounded, self-adjoint, and SO(2)-equivariant, \( V_L(\lambda_i) \) is a finite-dimensional, orthogonal SO(2)-representation. For \( \alpha > 0 \) define \( D_\alpha(\mathbb{H}) = \{ h \in \mathbb{H} : \|h\|_\mathbb{H} < \alpha \} \).

Combining Theorem 4.5 in [25] with Theorem 2.2 we obtain the following theorem.

**Theorem 2.4.** Under the above assumptions if \( 1 \not\in \sigma(L) \), then
\[
\nabla_{SO(2)} - \deg(Id - L, D_\alpha(\mathbb{H})) = \prod_{\lambda_i > 1} \nabla_{SO(2)} - \deg(-Id, D_\alpha(V_L(\lambda_i))).
\]

By \( C^1_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R}) \) we will denote the set of families of SO(2)-invariant \( C^1 \)-functionals. Let functional \( \Phi \in C^1_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R}) \) be such that
\[
\Phi(u, \lambda) = \frac{1}{2} \langle u - L(\lambda)u, u \rangle_\mathbb{H} + \eta(u, \lambda),
\]
where map \( \mathbb{H} \times \mathbb{R} \ni (u, \lambda) \to L(\lambda)(u) \in \mathbb{H} \) is compact and for any \( \lambda \in \mathbb{R} \) operator \( L(\lambda) : \mathbb{H} \to \mathbb{H} \) is linear, compact, self-adjoint, SO(2)-equivariant and \( \nabla_u \eta : \mathbb{H} \times \mathbb{R} \to \mathbb{H} \) is an SO(2)-equivariant compact operator and such that
\begin{itemize}
\item[a)] \( \nabla_u \eta(0, \lambda) = 0 \), for all \( \lambda \in \mathbb{R} \),
\item[b)] \( \nabla_u \eta(u, \lambda) = o(\|u\|) \), uniformly on bounded \( \lambda \)-intervals.
\end{itemize}

Put \( \mathcal{N}(\Phi) = \{(u, \lambda) \in (\mathbb{H} \setminus \{0\}) \times \mathbb{R} : \nabla_u \Phi(u, \lambda) = 0 \} \). Let \( C(\lambda_0) \) denote connected component of \( \text{cl}(\mathcal{N}(\Phi)) \) such that \((0, \lambda_0) \in C(\lambda_0) \).

**Definition 2.1.** A point \((0, \lambda_0) \in \mathbb{H} \times \mathbb{R} \) is said to be a bifurcation point of solutions of the equation \( \nabla_u \Phi(u, \lambda) = 0 \), if \((0, \lambda_0) \in \text{cl}(\mathcal{N}(\Phi)) \). A point \((0, \lambda_0) \in \mathbb{H} \times \mathbb{R} \) is said to be a branching point of solutions of the equation \( \nabla_u \Phi(u, \lambda) = 0 \), if \( C(\lambda_0) \setminus \{(0, \lambda_0)\} \neq \emptyset \).

Of course any branching point is a bifurcation point. It is worth to point out that there are bifurcation points which are not branching points.

**Remark 2.1.** Is is well-known that if \((0, \lambda_0) \in \mathbb{H} \times \mathbb{R} \) is a bifurcation point of solutions of the equation \( \nabla_u \Phi(u, \lambda) = 0 \), then the operator \( Id - L(\lambda_0) : \mathbb{H} \to \mathbb{H} \) is not an isomorphism.
Define $\mathcal{SP}(L) = \{ \lambda \in \mathbb{R} : Id - L(\lambda) \text{ is not an isomorphism} \}$ and assume that the set $\mathcal{SP}(L)$ does not possess accumulation points.

Fix $\lambda_0 \in \mathcal{SP}(L)$ and choose $\epsilon > 0$ such that $[\lambda_0 - \epsilon, \lambda_0 + \epsilon] \cap \mathcal{SP}(L) = \{ \lambda_0 \}$ and define a bifurcation index $\text{BIF}_{SO(2)}(\lambda_0, \nabla u \Phi) \in U(SO(2))$ as follows

$$\text{BIF}_{SO(2)}(\lambda_0, \nabla u \Phi) = \nabla_{SO(2)} - \text{deg}(Id - L(\lambda_0 + \epsilon), D_\alpha(\mathbb{H})) - \nabla_{SO(2)} - \text{deg}(Id - L(\lambda_0 - \epsilon), D_\alpha(\mathbb{H})).$$

The following theorem is a slight generalization version of the classical Rabinowitz global bifurcation theorem proved in [25] for the class of SO(2)-equivariant gradient operators. To prove this theorem we have used the infinite-dimensional version of the degree for SO(2)-equivariant gradient maps. In other words we study global bifurcations of critical points of SO(2)-invariant functionals. We formulate sufficient conditions for the existence of branching points of critical points of such functionals. Moreover, we study global properties of closed connected sets of critical points.

**Theorem 2.5.** Fix $\lambda_0 \in \mathcal{SP}(L)$ such that $\text{BIF}_{SO(2)}(\lambda_0, \nabla u \Phi) \neq \Theta \in U(SO(2)).$ Then

1. either $C(\lambda_0)$ is unbounded in $\mathbb{H} \times \mathbb{R}$,
2. or $C(\lambda_0)$ is bounded in $\mathbb{H} \times \mathbb{R}$ and additionally the following conditions are satisfied
   
   (a) $C(\lambda_0) \cap \{0\} \times \mathbb{R} = \{0\} \times \{\lambda_0, \lambda_1, \ldots, \lambda_p\} \subset \{0\} \times \mathcal{SP}(L),$
   
   (b) $\sum_{i=0}^{p} \text{BIF}_{SO(2)}(\lambda_i, \nabla u \Phi) = \Theta \in U(SO(2)).$

**Remark 2.2.** Suppose that the Hilbert space $(\mathbb{H}, \langle \cdot, \cdot \rangle_\mathbb{H})$ is not an orthogonal SO(2)-representation. Hence $C^1$-functional $\Phi$ is not SO(2)-invariant and consequently gradient map $\nabla \Phi : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ is not SO(2)-equivariant. Nevertheless, we can still prove non-equivariant version of Theorem 2.5 (so called the Rabinowitz global bifurcation theorem) using the Leray-Schauder degree instead of the degree for SO(2)-equivariant gradient maps, see for instance [21, 23, 24].

### 3. Linear Equation

In this section we study the following system of linear elliptic equations:

$$\begin{cases} -\Delta u = \Lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded subset with boundary of the class $C^{1-}$, $u = (u_1, \ldots, u_m)$ and $\Lambda$ is a symmetric $(m \times m)$-matrix.

#### 3.1. Generalized Eigenspaces and Eigenvalues ($m \in \mathbb{N}$).

Let us consider the Sobolev space $\mathbb{H} = \mathbb{H}^1_0(\Omega) \oplus \ldots \oplus \mathbb{H}^1_0(\Omega) = \bigoplus_{i=1}^{m} \mathbb{H}^1_0(\Omega)$ with the inner product defined as follows:

$$\langle v, w \rangle_\mathbb{H} = \sum_{i=1}^{m} \langle v_i, w_i \rangle_{\mathbb{H}^1_0(\Omega)} = \sum_{i=1}^{m} \int_{\Omega} \nabla v_i \nabla w_i, \text{ for } v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_m) \in \mathbb{H}.$$
Let us denote by \( \sigma(-\Delta; \Omega) = \{\lambda_k \in (0, +\infty); \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots\} \) the eigenvalues of the eigenvalue problem:
\[
\begin{align*}
-\Delta v &= \lambda v \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]
(3.1.1)

Let \( V_\Delta(\lambda_k) \) denote the eigenspace of \(-\Delta\) corresponding to the eigenvalue \( \lambda_k \in \sigma(-\Delta; \Omega) \) and put \( \mu_\Delta(\lambda_k) = \dim V_\Delta(\lambda_k) \).

It is known that the space \((\mathbb{H}, \langle \cdot, \cdot \rangle_\mathbb{H})\) is a separable Hilbert space and
\[
\mathbb{H} = \bigoplus_{k=1}^\infty \bigoplus_{j=1}^m V_\Delta(\lambda_k) = \bigoplus_{k=1}^\infty \mathbb{R}^m \otimes V_\Delta(\lambda_k).
\]

The study of solutions of system (3.1) is equivalent to the study of zeroes of an operator \( F = Id - L_\Lambda = Id - (-\Delta)^{-1}\Lambda : \mathbb{H} \rightarrow \mathbb{H} \), where operator \( L_\Lambda = (-\Delta)^{-1}\Lambda : \mathbb{H} \rightarrow \mathbb{H} \) is linear, compact and self-adjoint. Since subspaces \( \bigoplus_{j=1}^m V_\Delta(\lambda_k) \) are preserved by the operator \( F \),

it is enough to study \( F_k \) the restriction of \( F \) to \( \bigoplus_{j=1}^m V_\Delta(\lambda_k) \) for \( k \in \mathbb{N} \) i.e.

\[
F_k = Id - \lambda_k^{-1}\Lambda : \bigoplus_{j=1}^m V_\Delta(\lambda_k) \rightarrow \bigoplus_{j=1}^m V_\Delta(\lambda_k),
\]

for \( k \in \mathbb{N} \). Denoting by \( G : \mathbb{R}^m \otimes V_\Delta(\lambda_k) \rightarrow \bigoplus_{j=1}^m V_\Delta(\lambda_k) \) the natural isomorphism it is easy to check that for any \( k \in \mathbb{N} \) operator \( G^{-1} \circ F_k \circ G \) is defined as follows

\[
G^{-1} \circ F_k \circ G = (Id_{\mathbb{R}^m} - \lambda_k^{-1}\Lambda) \otimes Id_{V_\Delta(\lambda_k)} : \mathbb{R}^m \otimes V_\Delta(\lambda_k) \rightarrow \mathbb{R}^m \otimes V_\Delta(\lambda_k).
\]

Define \( \sigma(\Lambda) = \{\alpha_1, \ldots, \alpha_p\} \), where \( \sum_{j=1}^p \mu_\Lambda(\alpha_j) = m \). Since \( \mathbb{R}^m \otimes V_\Delta(\lambda_k) = \bigoplus_{j=1}^p V_\Lambda(\alpha_j) \otimes V_\Delta(\lambda_k) \), we have

\[
G^{-1} \circ F_k \circ G = \bigoplus_{j=1}^p \left(1 - \frac{\alpha_j}{\lambda_k} \right) Id_{V_\Lambda(\alpha_j)} \otimes Id_{V_\Delta(\lambda_k)}
\]

\[
: \bigoplus_{j=1}^p V_\Lambda(\alpha_j) \otimes V_\Delta(\lambda_k) \rightarrow \bigoplus_{j=1}^p V_\Lambda(\alpha_j) \otimes V_\Delta(\lambda_k).
\]

The following proposition is a direct consequence of the above computations. It will be extremely useful in the next sections. Part of this lemma has been proved in [4, 8] for \( n = 2 \). Denote by \( S(m; \mathbb{R}) \) the set of real, symmetric \((m \times m)\)-matrices and by \( O(m; \mathbb{R}) \) the set of real orthogonal \((m \times m)\)-matrices.

**Proposition 3.1.1.** Fix \( \Lambda \in S(m; \mathbb{R}) \). Then the following conditions are equivalent:

1. equation (3.1) possesses nonzero solution,
2. operator \( F \) is not an isomorphism,
Moreover, if \( \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \neq \emptyset \) then

\[
\begin{align*}
(1) \quad & \mathbb{H}_{(0,\Lambda)} = \ker F = V_{L_\Lambda}(1) = \bigoplus_{j=1}^{p} \bigoplus_{\alpha_j = \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; \Omega)} V_\Lambda(\alpha_j) \otimes V_{-\Delta}(\lambda_k), \\
(2) \quad & \mathbb{H}_{(-\Lambda)} = \bigoplus_{\gamma \in \sigma(L\Lambda) \cap (1, +\infty)} V_{L\Lambda}(\gamma) = \bigoplus_{j=1}^{p} \bigoplus_{\lambda_k \in (0, \alpha_j) \cap \sigma(-\Delta; \Omega)} V_\Lambda(\alpha_j) \otimes V_{-\Delta}(\lambda_k), \\
(3) \quad & \mathbb{H}_{(+\Lambda)} = (\mathbb{H}_0 \oplus \mathbb{H}_{-})^\perp.
\end{align*}
\]

**Proof.** (1) \( \iff \) (2) This equivalence is obvious. (2) \( \iff \) (3) The operator \( F \) is an isomorphism if and only if \( F_k \) is an isomorphism for any \( k \in \mathbb{N} \). On the other hand it is clear that \( F_k \) is an isomorphism iff \( \lambda_k \notin \sigma(\Lambda) \), which completes the proof. \( \square \)

3.2. **Generalized Eigenspaces and Eigenvalues \( (m = 2) \).** In this section we study the following system of linear elliptic equations:

\[
\begin{align*}
-\Delta v_1 &= \lambda v_1 + \delta v_2 & \text{in } \Omega, \\
-\Delta v_2 &= \delta v_1 + \gamma v_2 & \text{in } \Omega, \\
v_1 &= v_2 = 0 & \text{on } \partial \Omega.
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is an open, bounded subset with boundary of the class \( C^{1-} \).

Problem (3.2.1) can be written in the following way

\[
\begin{align*}
-\Delta v &= \Lambda v & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Lambda = \begin{bmatrix} \lambda & \delta \\ \delta & \gamma \end{bmatrix} \in S(2; \mathbb{R}) \).

To simplify computations we define a linear isomorphism \( \phi : S(2; \mathbb{R}) \to S(2; \mathbb{R}) \) as follows

\[
\phi \left( \begin{bmatrix} p_2 & p_1 \\ p_1 & p_3 \end{bmatrix} \right) = \Lambda_p := \begin{bmatrix} p_2 + p_3 & p_1 \\ p_1 & -p_2 + p_3 \end{bmatrix}.
\]

It is clear that equation (3.2.1) is equivalent to the following, more convenient in our considerations, problem

\[
\begin{align*}
-\Delta v_1 &= (p_2 + p_3)v_1 + p_1 v_2 & \text{in } \Omega, \\
-\Delta v_2 &= p_1 v_1 + (-p_2 + p_3)v_2 & \text{in } \Omega, \\
v_1 &= v_2 = 0 & \text{on } \partial \Omega.
\end{align*}
\]

Notice that we can rewrite equation (3.2.4) in following way

\[
\begin{align*}
-\Delta v &= \Lambda_p v & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
From now on we study system (3.2.5) instead of the equivalent system (3.2.2) and identify \( \mathbb{R}^3 \) with \( S(2; \mathbb{R}) \) by a map \( \psi : \mathbb{R}^3 \to S(2; \mathbb{R}) \) defined by
\[
\psi(p) = \Lambda_p.
\] (3.2.6)

Let us express eigenvalues of matrix \( \Lambda_p \) in terms of its elements, namely \( \sigma(\Lambda_p) = \{\alpha_{\pm}(p)\} \), where \( \alpha_{\pm}(p) = p_3 \mp \sqrt{p_1^2 + p_2^2} \).

**Remark 3.2.1.** From Proposition 3.1.1 we conclude that the operator \( \text{Id} - L_{\Lambda_p} \) is not an isomorphism iff there exists \( \lambda_k \in \sigma(-\Delta; \Omega) \) such that \( \lambda_k = \alpha_+(p) \) or \( \lambda_k = \alpha_-(p) \). Note, that \( \alpha_+(p) = \alpha_-(p) \) iff in \( p_1 = 0 \) and \( p_2 = 0 \).

Fix \( \lambda_k \in \sigma(-\Delta; \Omega) \) and define \( S_{\lambda_k}^+ = \{p \in \mathbb{R}^3, \alpha_+(p) = \lambda_k\}, S_{\lambda_k}^- = S_{\lambda_k}^+ \cup S_{\lambda_k}^- \). Moreover, put
\[
\sigma_{\text{even}}(-\Delta; \Omega) = \{\lambda_k \in \sigma(-\Delta; \Omega) : \mu_-(\lambda_k) \text{ is even}\},
\]
and denote
\[
S^\pm = \bigcup_{\lambda_k \in \sigma(-\Delta; \Omega)} S_{\lambda_k}^\pm, S = S^+ \cup S^-,
\]
\[
S^\pm_{\text{odd}} = \bigcup_{\lambda_k \in \sigma_{\text{odd}}(-\Delta; \Omega)} S_{\lambda_k}^\pm, S_{\text{odd}} = S^+_{\text{odd}} \cup S^-_{\text{odd}},
\]
\[
S^\pm_{\text{even}} = \bigcup_{\lambda_k \in \sigma_{\text{even}}(-\Delta; \Omega)} S_{\lambda_k}^\pm, S_{\text{even}} = S^+_{\text{even}} \cup S^-_{\text{even}}.
\]

**Remark 3.2.2.** Fix \( \lambda_k \in \sigma(-\Delta; \Omega) \). Then \( S_{\lambda_k}^\pm \) is a cone touching \( p_3 \)-axis at \( (0, 0, \lambda_k) \in \mathbb{R}^3 \). Moreover, the operator \( \text{Id} - L_{\Lambda_p} : \mathbb{H} \to \mathbb{H} \) is not an isomorphism iff \( p_0 \in S \).

In the lemma below we describe \( \ker(\text{Id} - L_{\Lambda_p}) \) in terms of eigenspaces of \(-\Delta\).

**Lemma 3.2.1.** Fix \( p_0 \in S \). Then,

1. if \( \sigma(-\Delta; \Omega) \cap \sigma(\Lambda_{p_0}) = \{\lambda_k\} \) and \( \mu_{\Lambda_{p_0}}(\lambda_k) = 1 \), then
   - (a) \( p_0 \in S_{\lambda_k}^+ \setminus \left( \bigcup_{k \neq k'} S_{\lambda_{k'}} \cup \{(0, 0, \lambda_k)\} \right) \),
   - (b) \( \mathbb{H}_{(0, \Lambda_{p_0})} = V_{\Lambda_{p_0}}(\lambda_k) \otimes V_\Delta(\lambda_k) \),
   - (c) \( \dim \mathbb{H}_{(0, \Lambda_{p_0})} = \mu_\Delta(\lambda_k) \);
2. if \( \sigma(-\Delta; \Omega) \cap \sigma(\Lambda_{p_0}) = \{\lambda_k\} \) and \( \mu_{\Lambda_{p_0}}(\lambda_k) = 2 \), then
   - (a) \( p_0 \in S_{\lambda_k}^+ \cap S_{\lambda_k}^- = \{(0, 0, \lambda_k)\} \),
   - (b) \( \mathbb{H}_{(0, \Lambda_{p_0})} = \mathbb{R}^2 \otimes V_\Delta(\lambda_k) \),
   - (c) \( \dim \mathbb{H}_{(0, \Lambda_{p_0})} = 2 \cdot \mu_\Delta(\lambda_k) \);
3. if \( \sigma(-\Delta; \Omega) \cap \sigma(\Lambda_{p_0}) = \{\lambda_k, \lambda_{k'}\} \) and \( k < k' \), then
   - (a) \( p_0 \in S_{\lambda_k}^+ \cap S_{\lambda_{k'}}^- \),
   - (b) \( \mathbb{H}_{(0, \Lambda_{p_0})} = V_{\Lambda_{p_0}}(\lambda_k) \otimes V_\Delta(\lambda_k) \oplus V_{\Lambda_{p_0}}(\lambda_{k'}) \otimes V_\Delta(\lambda_{k'}) \),
   - (c) \( \dim \mathbb{H}_{(0, \Lambda_{p_0})} = \mu_\Delta(\lambda_k) + \mu_\Delta(\lambda_{k'}) \).
Proof. (1) Suppose contrary to our claim that \( p_0 \in S_{\lambda_k} \cap \bigcup_{k \neq k'} S_{\lambda_{k'}} \) or \( p_0 = (0,0,\lambda_k) \). If \( p_0 \in S_{\lambda_k} \cap \bigcup_{k \neq k'} S_{\lambda_{k'}} \), then \( \sigma(-\Delta; \Omega) \cap \sigma(\Lambda_{p_0}) = \{ \lambda_k, \lambda_{k'} \}, k \neq k' \), a contradiction. If \( p_0 = (0,0,\lambda_k) \), then \( \sigma(-\Delta; \Omega) \cap \sigma(\Lambda_{p_0}) = \{ \lambda_k \} \) and \( \mu_{\Lambda_{p_0}}(\lambda_k) = 2 \), a contradiction.

Equality \( \mathbb{H}_{(0,\Lambda_{p_0})} = V_{\Lambda_{p_0}}(\lambda_k) \otimes V_{-\Delta}(\lambda_k) \) is a consequence of Lemma 3.1.1. (2) Since \( \Lambda_{p_0} \) is a \((2 \times 2)\)-matrix, \( \lambda_k \in \sigma(\Lambda_{p_0}) \) and \( \mu_{\Lambda_{p_0}}(\lambda_k) = 2 \), \( \Lambda_{p_0} = \begin{bmatrix} \lambda_k & 0 \\ 0 & \lambda_k \end{bmatrix} \). Equality

\[
\mathbb{H}_{(0,\Lambda_{p_0})} = \mathbb{H} \cap V_{-\Delta}(\lambda_k) \times V_{\Delta}(\lambda_k)
\]

is a consequence of Lemma 3.1.1. □

3.3. Bifurcation index \((m \in \mathbb{N})\). In this section we treat \( \mathbb{R}^n \) as an orthogonal \( \text{SO}(2) \)-representation and assume additionally that \( \Omega \subset \mathbb{R}^n \) is \( \text{SO}(2) \)-invariant. Hence \((\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}})\) is an orthogonal \( \text{SO}(2) \)-representation with \( \text{SO}(2) \)-action given by the formula \( (g \cdot v)(x) = v(gx) \) for any \( g \in \text{SO}(2) \) and \( v \in \mathbb{H} \). In this situation the operator \( \text{Id} - L_{\Lambda} : \mathbb{H} \rightarrow \mathbb{H} \) is \( \text{SO}(2) \)-equivariant and self-adjoint therefore we derive formula for bifurcation indices in terms of the degree for \( \text{SO}(2) \)-equivariant gradient operators.

Define

\[
\widehat{S}(m; \mathbb{R}) = \{ \Lambda \in S(m; \mathbb{R}) : \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \neq \emptyset \},
\]

\[
\widehat{S}_{\text{odd}}(m; \mathbb{R}) = \{ \Lambda \in \widehat{S}(m; \mathbb{R}) : \sigma(\Lambda) \cap \sigma_{\text{odd}}(-\Delta; \Omega) \neq \emptyset \},
\]

\[
\widehat{S}_{\text{even}}(m; \mathbb{R}) = \widehat{S}(m; \mathbb{R}) \setminus \widehat{S}_{\text{odd}}(m; \mathbb{R}).
\]

We call \( \widehat{S}(m; \mathbb{R}) \) the set of generalized eigenvalues of system (3.1).

The following two lemmas are direct consequence of Lemma 3.1.1.

**Lemma 3.3.1.** Fix \( \Lambda \in S(m; \mathbb{R}) \setminus \widehat{S}(m; \mathbb{R}) \). Then

\[
\nabla_{\text{SO}(2)} - \deg(\text{Id} - L_{\Lambda}, D_\alpha(\mathbb{H}), 0) = \nabla_{\text{SO}(2)} - \deg(-\text{Id}, D_\alpha(\mathbb{H}(-,\Lambda))).
\]

**Lemma 3.3.2.** Fix \( \Lambda \in \widehat{S}(m; \mathbb{R}) \). Then there is \( \epsilon_0 > 0 \) such that for any \( \epsilon > \epsilon_0 \) the following equalities hold true:

\[
\nabla_{\text{SO}(2)} - \deg(\text{Id} - L_{\Lambda + \epsilon \text{Id}}, D_\alpha(\mathbb{H}), 0) = \nabla_{\text{SO}(2)} - \deg(-\text{Id}, D_\alpha(\mathbb{H}(-,\Lambda))) \star \nabla_{\text{SO}(2)} - \deg(-\text{Id}, D_\alpha(\mathbb{H}(-,\Lambda))),
\]

\[
\nabla_{\text{SO}(2)} - \deg(\text{Id} - L_{\Lambda - \epsilon \text{Id}}, D_\alpha(\mathbb{H}), 0) = \nabla_{\text{SO}(2)} - \deg(-\text{Id}, D_\alpha(\mathbb{H}(-,\Lambda))).
\]

Fix \( \Lambda \in \widehat{S}_{\text{even}}(m; \mathbb{R}) \) and define a bifurcation index \( \mathcal{BIF}(\Lambda) \in U(\text{SO}(2)) \) as follows

\[
\mathcal{BIF}_{\text{SO}(2)}(\Lambda) = \nabla_{\text{SO}(2)} - \deg(-\text{Id}, D_\alpha(\mathbb{H}(-,\Lambda))) \star (\nabla_{\text{SO}(2)} - \deg(-\text{Id}, D_\alpha(\mathbb{H}(0,\Lambda))) - \mathbb{I})
\]

(3.3.1)
**Definition 3.3.1.** A matrix $\Lambda \in \widehat{S}_{\text{even}}(m; \mathbb{R})$ is called SO(2)-essential if $BIF_{SO(2)}(\Lambda) \neq \emptyset \in U(SO(2))$. Denote by $\widehat{S}_{\text{even}}(m; \mathbb{R})$ the set of SO(2)-essential matrices.

**Lemma 3.3.3.** A matrix $\Lambda \in \widehat{S}_{\text{even}}(m; \mathbb{R})$ is SO(2)-essential if and only if $\nabla_\Theta$ is even-dimensional nontrivial SO(2)-representation.

**Proof.** First of all, taking into account (2.2) and Lemma 2.1, we obtain $BIF_{SO(2)}(\Lambda) \neq \emptyset$ if and only if $\nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{(0,\Lambda)})) - \mathbb{I} \neq \emptyset \in U(SO(2))$. Since $\Lambda \in \widehat{S}_{\text{even}}(m; \mathbb{R})$, dim $\mathbb{H}_{(0,\Lambda)}$ is even. In view of Lemma 2.1 we have $\nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{(0,\Lambda)})) \neq \emptyset$ if and only if $\mathbb{H}_{(0,\Lambda)}$ is a nontrivial SO(2)-representation, which completes the proof. □

**Remark 3.3.1.** In other words $\Lambda \in \widehat{S}_{\text{even}}(m; \mathbb{R})$ is SO(2)-essential if and only if there is $\lambda_k \in \sigma(\Lambda) \cap \sigma_{\text{even}}(-\Delta; \Omega)$ such that $V_{-\Delta}(\lambda_k)$ is an even-dimensional nontrivial SO(2)-representation.

### 3.4. Bifurcation index ($m = 2$).

The following lemma is a direct consequence of Lemma 3.1.1. Fix $p_0 \not\in S$ and for $\alpha_\pm(p_0) \in \sigma(\Lambda_{p_0})$ define $Q(\alpha_\pm(p_0)) = \mathbb{R}[2,0] \bigoplus \bigoplus_{\lambda \prec \alpha_\pm(p_0)} V_{-\Delta}(\lambda)$. It is understood that $Q(\alpha_\pm(p_0)) = \mathbb{R}[2,0]$ for $\alpha_\pm(p_0) < \lambda_1$.

**Lemma 3.4.1.** If $p_0 \not\in S$ then

$$
\nabla_{SO(2)} \deg(-Id - L_{\Lambda_{p_0}}, D_\alpha(\mathbb{H})) = \\
= \nabla_{SO(2)} \deg(-Id, D_\alpha(Q(\alpha_+(p_0)))) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(Q(\alpha_-(p_0))))
$$

In the following lemma we compute the bifurcation index $BIF(\Lambda_{p_0}) \in U(SO(2))$. For $k \in \mathbb{N}$ define $\mathbb{H}_k = \mathbb{R}[2,0] \bigoplus \bigoplus_{i=1}^k V_{-\Delta}(\lambda_i)$. It is understood that $\mathbb{H}_k = \mathbb{R}[2,0]$ for $k < 1$.

**Lemma 3.4.2.** Fix $p_0 \in S_{\text{even}}$. Then

1. if $p_0 \in S_{\text{even}}^+ \cap S_{\text{even}}^-$ then there are $k_1, k_2 \in \mathbb{N}$ such that $k_2 \geq k_1, \alpha_+(p_0) = \lambda_{k_1}$ and $\alpha_-(p_0) = \lambda_{k_2}$ and we obtain

   $$
   BIF_{SO(2)}(\Lambda_{p_0}) = \\
   = \nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{k_1-1})) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{k_2-1})) \ast
   $$$$\ast \nabla_{SO(2)} \deg(-Id, D_\alpha(V_{-\Delta}(\lambda_{k_1}))) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(V_{-\Delta}(\lambda_{k_2}))) - \mathbb{I}),
   $$

2. if $p_0 \in S_{\text{even}}^+ \setminus S_{\text{even}}^-$ then there are $k_1, k_2 \in \mathbb{N} (k_2 > k_1)$ such that $\alpha_+(p_0) = \lambda_{k_1}$ and $\alpha_-(p_0) = \lambda_{k_2}$ and we obtain

   $$
   BIF_{SO(2)}(\Lambda_{p_0}) = \nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{k_1-1})) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{k_2}) \ast
   $$$$\ast \nabla_{SO(2)} \deg(-Id, D_\alpha(V_{-\Delta}(\lambda_{k_1}))) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(V_{-\Delta}(\lambda_{k_2}))) - \mathbb{I}),
   $$

3. if $p_0 \in S_{\text{even}}^- \setminus S_{\text{even}}^+$ then there are $k_1, k_2 \in \mathbb{N} (k_2 \geq k_1)$ such that $\alpha_-(p_0) = \lambda_{k_2}$ and $\alpha_+(p_0) = \lambda_{k_1}$ and $\lambda_{k_1-1} < \lambda_-(p_0) < \lambda_{k_2+1}$ and we obtain

   $$
   BIF_{SO(2)}(\Lambda_{p_0}) = \nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{k_1-1})) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(\mathbb{H}_{k_2}) \ast
   $$$$\ast \nabla_{SO(2)} \deg(-Id, D_\alpha(V_{-\Delta}(\lambda_{k_1})))) \ast \nabla_{SO(2)} \deg(-Id, D_\alpha(V_{-\Delta}(\lambda_{k_2}))) - \mathbb{I}).
   $$
Proof. (1) By Lemma 3.4.1 we obtain

$$\nabla_{SO(2)} - \text{deg}(Id - L_{\Lambda(p_0)} + \epsilon Id, D_\alpha(\mathbb{H})) =$$

$$= \nabla_{SO(2)} - \text{deg}\left(-Id, D_\alpha(Q(\beta_+(p_0)))\right) \ast \nabla_{SO(2)} - \text{deg}\left(-Id, D_\alpha(Q(\beta_-(p_0)))\right),$$

where $\beta_\pm(p_0) = \alpha_\pm(p_0) + \epsilon$. Thus we conclude that

$$\nabla_{SO(2)} - \text{deg}(Id - L_{\Lambda(p_0)} + \epsilon Id, D_\alpha(\mathbb{H})) =$$

$$= \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_1})) \ast \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_2})).$$

Similarly, by Lemma 3.4.1 we obtain

$$\nabla_{SO(2)} - \text{deg}(Id - L_{\Lambda(p_0)} - \epsilon Id, D_\alpha(\mathbb{H})) =$$

$$= \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(Q(\delta_+(p_0)))) \ast \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(Q(\delta_-(p_0)))) =$$

where $\delta_\pm(p_0) = \alpha_\pm(p_0) - \epsilon$.

Therefore

$$\nabla_{SO(2)} - \text{deg}(Id - L_{\Lambda(p_0)} - \epsilon Id, D_\alpha(\mathbb{H})) =$$

$$= \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_1-1})) \ast \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_2-1})).$$

Combining (3.3.1) with (3.4.1) and (3.4.2) we complete our proof.

(2). To complete the proof it is sufficient to note that

$$\nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(Q(\beta_-(p_0)))) = \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(Q(\delta_-(p_0)))) =$$

$$= \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_2})).$$

and

$$\nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(Q(\beta_+(p_0)))) = \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_1})).$$

and

$$\nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(Q(\delta_+(p_0)))) = \nabla_{SO(2)} - \text{deg}(-Id, D_\alpha(\mathbb{H}_{k_1-1})).$$

where $\delta_\pm(p_0) = \alpha_\pm(p_0) - \epsilon$, $\beta_\pm(p_0) = \alpha_\pm(p_0) + \epsilon$.

Prove of (3) is in fact the same as the prove of (2). \qed

It is of interest to formulate sufficient and necessary conditions for the nontriviality of the bifurcation index. In the following lemma we study the bifurcation index.

**Lemma 3.4.3.** Fix $p_0 \in S_{\text{even}}$. Then

(1) If $p_0 \in S_{\text{even}}^+ \cap S_{\text{even}}^-$ then there are $k_1, k_2 \in \mathbb{N}(k_2 \geq k_1)$ such that $\alpha_+(p_0) = \lambda_{k_1}$, $\alpha_-(p_0) = \lambda_{k_2}$. Moreover, a matrix $\Lambda_{p_0}$ is essential if and only if $V_\Delta(\lambda_{k_1}) \oplus V_\Delta(\lambda_{k_2})$ is a nontrivial $SO(2)$-representation,
Define a CUSP proof. (1) From Lemma (3.4.2), (2.2) and (2.3) we obtain that
\[ \alpha_+(p_0) = \lambda_{k_1} \text{ and } \lambda_{k_2} < \alpha_-(p_0) < \lambda_{k_2+1}. \]
Moreover, a matrix \( \Lambda_{p_0} \) is essential if and only if \( V_-(\lambda_{k_1}) \) is a nontrivial \( SO(2) \)-representation.

(2) If \( p_0 \in S_{even}^+ \setminus S_{even}^- \) then there are \( k_1, k_2 \in \mathbb{N} \) \((k_2 > k_1)\) such that \( \alpha_+(p_0) = \lambda_{k_1} \) and \( \lambda_{k_2} < \alpha_-(p_0) < \lambda_{k_2+1} \). Moreover, a matrix \( \Lambda_{p_0} \) is essential if and only if \( V_-(\lambda_{k_1}) \) is a nontrivial \( SO(2) \)-representation.

(3) If \( p_0 \in S_{even}^+ \setminus S_{even}^- \) then there are \( k_1, k_2 \in \mathbb{N} \) \((k_2 \geq k_1)\) such that \( \alpha_-(p_0) = \lambda_{k_2} \) and \( \lambda_{k_2-1} < \alpha_+(p_0) < \lambda_{k_1}(\lambda_0 = -\infty) \). Moreover, a matrix \( \Lambda_{p_0} \) is essential if and only if \( V_-(\lambda_{k_1}) \) is a nontrivial \( SO(2) \)-representation.

Proof. (1) From Lemma (3.4.2), (2.2) and (2.3) we obtain that \( BF(\Lambda_{p_0}) \neq \emptyset \in U(SO(2)) \) if and only if
\[ \nabla_{SO(2)} \deg(-Id, D_\alpha(V_-(\lambda_{k_1}) \oplus V_-(\lambda_{k_2}))) \neq I \in U(SO(2)). \]  
(3.4.3)

Since \( p_0 \in S_{even} \), \( \dim V_-(\lambda_{k_1}) \oplus V_-(\lambda_{k_2}) \) is even. Therefore condition (3.4.3) is satisfied if and only if \( V_-(\lambda_{k_1}) \oplus V_-(\lambda_{k_2}) \) is a nontrivial \( SO(2) \)-representation, which completes the proof. Proofs of (2) and (3) are in fact the same as the proof of (1). \( \square \)

Define
\[ \sigma_{SO(2)}(-\Delta; \Omega) = \{ \lambda_k \in \sigma_{even}(-\Delta; \Omega) : V_-(\lambda_k) \text{ is a nontrivial } SO(2) \text{-representation} \}. \]

The following corollary is a direct consequence of Lemma 3.4.3.

**Corollary 3.4.1.** If \( \lambda_k \in \sigma_{SO(2)}(-\Delta; \Omega) \) and \( p_0 \in S_{\lambda_k} \setminus S_{odd} \) then matrix \( \Lambda_{p_0} \) is \( SO(2) \)-essential.

## 4. Nonlinear Equation

In this section we study solutions of the following system of differential equations
\[
\begin{cases}
-\Delta u = \nabla u F(u, \Lambda) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  
(4.1)

where

1. \( \Omega \subset \mathbb{R}^n \) is an open, bounded subset with \( C^1 \)-boundary,
2. \( u = (u_1, \ldots, u_m) \),
3. \( F \in C^2(\mathbb{R}^m \times S(m; \mathbb{R}), \mathbb{R}) \) satisfies the following conditions
   (a) \( F(x, \Lambda) = \frac{1}{2} \langle \Lambda x, x \rangle + \eta(x, \Lambda) \),
   (b) for any \( \Lambda \in S(m; \mathbb{R}) \) there are \( C_\Lambda > 0 \) and \( 1 \leq p_\Lambda < (N + 2)(N - 2)^{-1} \) such that
      \[ |\nabla u F(u, \Lambda)| \leq C_\Lambda (1 + |u|)^{p_\Lambda}, \]
   (c) \( \nabla_x \eta(0, \Lambda) = 0 \) for any \( \Lambda \in S(m; \mathbb{R}) \),
   (d) \( \nabla_x^2 \eta(0, \Lambda) = 0 \) for any \( \Lambda \in S(m; \mathbb{R}) \).

Define a \( C^2 \)-functional \( \Phi : H \times S(m; \mathbb{R}) \rightarrow \mathbb{R} \) as follows
\[
\Phi(u, \Lambda) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 - (\Lambda u(x), u(x)) dx - \int_\Omega \eta(u(x), \Lambda) dx.
\]  
(4.2)
It is known that the weak solutions of system (4.1) are in one-to-one correspondence to the critical points of functional $\Phi$ (with respect to $u$). Therefore to study weak solutions of system (4.1) it is enough to study critical points of the functional $\Phi$. It is known that

$$\nabla_u \Phi(u, \Lambda) = (Id - L_\Lambda)u - \nabla N(u, \Lambda),$$

(4.3)

where

- $L_\Lambda = (-\Delta)^{-1}\Lambda : H \to H$ is a linear, self-adjoint, bounded and compact operator,
- $\nabla N : H \times \mathbb{R} \to H$ is a gradient, compact operator of class $C^1$ such that $\nabla N(u, p) = \sigma(|u|)$ for $u = 0$, uniformly for $\lambda$ in bounded subsets of $S(m; \mathbb{R})$.

It is clear that $\nabla_u^2 \Phi(0, \Lambda) = Id - L_\Lambda : H \to H$.

Put $N(\nabla_u \Phi) = \{(u, p) \in (H \setminus \{0\}) \times S(m; \mathbb{R}) : \nabla_u \Phi(u, \Lambda) = 0\}$. Let $C(\Lambda)$ denote connected component of $cl(N(\nabla_u \Phi))$ such that $(0, \Lambda) \in C(\Lambda)$.

**Definition 4.1.** A point $(0, \Lambda_0) \in H \times S(m; \mathbb{R})$ is said to be a bifurcation point of weak solutions of (4.1) if $(0, \Lambda_0) \in cl(N(\nabla_u \Phi))$. The set of bifurcation points of (4.1) will be denoted by $\mathcal{B}(\nabla_u \Phi)$. A point $(0, \Lambda_0) \in H \times S(m; \mathbb{R})$ is said to be a branching point of weak solutions of (4.1), if $C(\Lambda_0) \setminus \{(0, \Lambda_0)\} \neq \emptyset$. The set of branching points of solutions of (4.1) will be denoted by $\mathcal{B}(\nabla_u \Phi)$. A point $(0, \Lambda_0) \in H \times S(m; \mathbb{R})$ is said to be a global bifurcation point of weak solutions of (4.1), if either $C(\Lambda_0)$ is unbounded or $(C(\Lambda_0) \setminus \{(0, \Lambda_0)\}) \cap \{(0) \times S(m; \mathbb{R})\} \neq \emptyset$. The set of global bifurcation points of solutions of (4.1) will be denoted by $\mathcal{G}(\nabla_u \Phi)$.

**Remark 4.1.** It is known that if a point $(0, \Lambda_0) \in H \times S(m; \mathbb{R})$ is a bifurcation point of solutions of (4.1), then $\nabla_u^2 \Phi(0, \Lambda_0) = Id - L_\Lambda$ is not an isomorphism (or equivalently by Proposition 3.1.1 $\sigma(\Lambda_0) \cap \sigma(-\Delta; \Omega) \neq \emptyset$). Moreover, it is clear that $\mathcal{G}(\nabla_u \Phi) \subseteq \mathcal{B}(\nabla_u \Phi) \subseteq \mathcal{G}(\nabla_u \Phi) \subset \mathcal{S}(m; \mathbb{R})$.

4.1. **Bifurcation theorems** ($m \in \mathbb{N}$). We start studies of weak solutions of system (4.1) with local bifurcation theorems. In the following lemma we reduce a local problem of studies of the set $(\nabla_u \Phi)^{-1}(0)$ in a neighborhood of $(0, \Lambda_0) \in H \times S(m; \mathbb{R})$ to a finite-dimensional one. Namely, we apply the Liapunov-Schmidt reduction. Denote by $\pi_\Lambda : H \to H$ the orthogonal projection on $H_{(0,\Lambda_0)}$.

**Lemma 4.1.1.** Fix $\Lambda_0 \in \mathcal{S}(m; \mathbb{R})$ such that $\sigma(\Lambda_0) \cap \sigma(-\Delta; \Omega) = \{\lambda_k_1, \ldots, \lambda_k_p\}$ and define an operator $\Psi : H \times \mathbb{R} \to \mathbb{R}$ by $\Psi(u, \epsilon) = \Phi(u, \Lambda_0 + \epsilon Id_{\mathbb{R}^m})$. Then, there are

1. an open neighborhood $\mathcal{V}_0 \subset H_{(0,\Lambda_0)} \times \mathbb{R}$ of $(0, 0)$,
2. a $C^1$-function $w : \mathcal{V}_0 \to H_{(0,\Lambda_0)}$ satisfying
   - (a) $w(0, \epsilon) = 0$ for all $(0, \epsilon) \in \mathcal{V}_0$,
   - (b) $w'(0, \epsilon) = 0$ for all $(0, \epsilon) \in \mathcal{V}_0$.

such that the only solutions of equation $\nabla_u \Psi(u, \epsilon) = 0$ close to $(0, 0) \in H \times \mathbb{R}$ are in one to one correspondence with solutions of the following bifurcation equation

$$\nabla_u G(v, \epsilon) = 0,$$

(4.4.1)

where potential $G : \mathcal{V}_0 \to \mathbb{R}$ is such that $G(v, \epsilon) = \Psi(v, w(v, \epsilon), \epsilon)$.
Moreover,
\[ \nabla_v G(v, \epsilon) = \mathcal{L}(v, \epsilon) - \pi_{\Lambda_0} \left( \nabla N(v, w(v, \epsilon), \epsilon) \right) = 
-\epsilon ((-\Delta)^{-1} I_{\mathbb{R}^m})(v) - \pi_{\Lambda_0} \left( \nabla N(v, w(v, \epsilon), \epsilon) \right). \]

Additionally, we have \( \mathbb{H}_{(0, \Lambda_0)} = \bigoplus_{i=1}^{p} V_{\lambda} (\lambda_{k_i}) \otimes V_\Delta (\lambda_{k_i}) \) and
\[ \mathcal{L}(v_{k_1}, \ldots, v_{k_p}, \epsilon) = -\epsilon (\lambda_{k_1}^{-1} v_{k_1}, \ldots, \lambda_{k_p}^{-1} v_{k_p}). \] (4.1.2)

**Proof.** We are going to study weak solutions of system (4.1) in a neighborhood of \((0, 0) \in \mathbb{H} \times \mathbb{R} \). Therefore from now on we study solutions of equation
\[ \nabla_u \Psi(u, \epsilon) = 0. \] (4.1.3)
Equation (4.1.3) is equivalent to the following system of equations
\[ F_1(u, \lambda) := (I_{\theta} - \pi_{\Lambda_0}) (\nabla_u \Psi(u, \epsilon)) = 0, \]
\[ F_2(u, \epsilon) := \pi_{\Lambda_0} (\nabla_u \Psi(u, \epsilon)) = 0. \] (4.1.4)

Let \( u = (v, w) \in \mathbb{H} = \ker(I_{\theta} - \pi_{\Lambda_0}) \oplus \text{im} (I_{\theta} - \pi_{\Lambda_0}) \). Notice that
1. \( F_1(0, 0) = 0 \),
2. \( D_u F_1(0, 0) = I_{\theta} - L_{\Lambda_0} : \mathbb{H}^+_{(0, \Lambda_0)} \to \mathbb{H}^+_{(0, \Lambda_0)} \) is an isomorphism.

By the implicit function theorem we obtain that there are open sets \( \mathcal{V}_0 \subset \ker(I_{\theta} - \pi_{\Lambda_0}) \times \mathbb{R}, \mathcal{V}_1 \subset \text{im} (I_{\theta} - \pi_{\Lambda_0}) \) and \( C^1 \)-function \( w : \mathcal{V}_0 \to \mathcal{V}_1 \) such that
1. \((0, 0) \in \mathcal{V}_0 \) and \( 0 \in \mathcal{V}_1 \),
2. \( F_1(v, w, \epsilon) = 0 \) and \((v, \epsilon, w) \in \mathcal{V}_0 \times \mathcal{V}_1 \) if \( w = w(v, \epsilon) \),
3. \( w(0, \epsilon) = 0 \) for all \((0, \epsilon) \in \mathcal{V}_0 \),
4. \( w'(0, \epsilon) = 0 \) for all \((0, \epsilon) \in \mathcal{V}_0 \).

Summing up, in other to study solutions of equation (4.1.3) in \( \mathcal{V}_0 \times \mathcal{V}_1 \) it is enough to study the following equation \( F_2(v, w(v, \epsilon), \epsilon) = 0 \) for \((v, \epsilon) \in \mathcal{V}_0 \).

Notice that
\[ F_2(v, w(v, \epsilon), \epsilon) = \pi_{\Lambda_0} (\nabla_u \Psi(v, w(v, \epsilon), \epsilon)) = \]
\[ = \pi_{\Lambda_0} ((v, w(v, \epsilon)) - L_{\Lambda_0} (v, w(v, \epsilon))) + \]
\[ + \pi_{\Lambda_0} (L_{\Lambda_0} (v, w(v, \epsilon)) - L_{\Lambda_0 + \epsilon I_{\mathbb{R}^m}} (v, w(v, \epsilon)) - \nabla N(v, w(v, \epsilon), \Lambda_0 + \epsilon I_{\mathbb{R}^m})) = \]
\[ = \pi_{\Lambda_0} (L_{\Lambda_0} (v, w(v, \epsilon)) - L_{\Lambda_0 + \epsilon I_{\mathbb{R}^m}} (v, w(v, \epsilon)) - \nabla N(v, w(v, \epsilon), \Lambda_0 + \epsilon I_{\mathbb{R}^m})) = \]
\[ = \pi_{\Lambda_0} (L_{(-\epsilon I_{\mathbb{R}^m}} (v, w(v, \epsilon)) - \nabla N(v, w(v, \epsilon), \Lambda_0 + \epsilon I_{\mathbb{R}^m})) = \]
\[ = L_{-\epsilon I_{\mathbb{R}^m}} (v) - \pi_{\Lambda_0} (\nabla N(v, w(v, \epsilon), \Lambda_0 + \epsilon I_{\mathbb{R}^m})) = \]
Let \( -\epsilon((\Delta)^{-1} I_{\mathbb{R}^m})(v) - \pi_{\Lambda_0} (\nabla N(v, w(v, \epsilon), \Lambda_0 + \epsilon I_{\mathbb{R}^m})) \), which completes the proof.

In the following theorem we describe the set of bifurcation points of solutions of \((4.1)\). We prove this theorem applying the finite-dimensional Liapunov-Schmidt reduction and the Conley index.

**Theorem 4.1.1.** Under the above assumptions: \( \text{BIF}(\nabla \Phi) = \{0\} \times \hat{S}(m; \mathbb{R}). \)

**Proof.** From Remark 4.1 it follows that \( \text{BIF}(\nabla \Phi) \subset \{0\} \times \hat{S}(m; \mathbb{R}). \) Fix \((0, \Lambda_0) \in \{0\} \times \hat{S}(m; \mathbb{R}) \subset \mathbb{H} \times \hat{S}(m; \mathbb{R}). \) What is left is to show that \((0, \Lambda_0) \in \text{BIF}(\nabla \Phi). \) We are going to study the set of solutions \((4.1.3)\) in a neighborhood of \((0, \Lambda_0) \in \mathbb{H} \times \hat{S}(m; \mathbb{R}). \) By Lemma 4.1.1 it is enough to study solutions of the following equation

\[
\nabla_v G(v, \epsilon) = L(v, \epsilon)(v) + h.o.t. = 0
\]

where \((v, \epsilon) \in \mathbb{H}_{(0, \Lambda_0)} \times \mathbb{R} \) is sufficiently small and \( L \) is given by \((4.1.2)\). Since all the elements of \( \sigma(-\Delta; \Omega) \) are nonzero, \( L(\epsilon, \epsilon) \) is an isomorphism for any \( \epsilon \neq 0 \). Therefore for any fixed \( \epsilon \neq 0 \) the origin \( \{0\} \subset \mathbb{H}_{(0, \Lambda_0)} \) is an isolated invariant set of a flow generated by the gradient map given by \((4.1.5)\). Fix sufficiently small \( \epsilon > 0 \). Since all the elements of \( \sigma(-\Delta; \Omega) \) are positive one can compute the Conley index \( CI(\{0\}, \nabla_v G(\epsilon, \epsilon)) \) of \( \{0\} \subset \mathbb{H}_{(0, \Lambda_0)} \). Namely,

\[
CI(\{0\}, \nabla_v G(\epsilon, \epsilon)) = \begin{cases} 
S^\dim H_{(0, \Lambda_0)} & \text{if } \epsilon < 0, \\
S^0 & \text{if } \epsilon > 0.
\end{cases}
\]

It is known that change of the Conley index implies the existence of a bifurcation point of zeroes of a family of gradient maps, see for instance [17]. Since \((4.1.6)\), \((0, 0) \in \mathbb{H}_{(0, \Lambda_0)} \times \mathbb{R} \) is a bifurcation point of solutions of the equation \((4.1.5)\). Applying once more Lemma 4.1.1 we obtain that \((0, \Lambda_0) \in \text{BIF}(\nabla \Phi), \) which completes the proof.

The following theorem has been proved in non-variational case and \( m = 2 \) in [12], see Theorem 2.1 of [12]. In fact the idea of this theorem is the same as that of Theorem 2.1 of [12]. To prove this theorem we apply the Leray-Schauder degree.

**Theorem 4.1.2.** Under the assumptions of Theorem 4.1.1:

\[
\{0\} \times \hat{S}_{\text{odd}}(m; \mathbb{R}) \subset \mathcal{GLOB}(\nabla \Phi).
\]

**Proof.** Fix \( \Lambda_0 \in \hat{S}_{\text{odd}}(m; \mathbb{R}). \) Let \( J(\Lambda_0) = \text{diag} \{\alpha_1, \ldots, \alpha_m\} \) be the Jordan normal form of \( \Lambda_0 \) and let \( P \in O(m; \mathbb{R}) \) be such that \( \Lambda_0 = PJ(\Lambda_0)P^{-1}. \) Since \( \Lambda_0 \in \hat{S}_{\text{odd}}(m; \mathbb{R}) \) there is \( \alpha_{j_0} = \lambda_k \in \sigma(\Lambda) \cap \sigma_{\text{odd}}(-\Delta; \Omega). \) For any \( \epsilon \in \mathbb{R} \) define a matrix \( A_{0, \epsilon} = PJ(\Lambda_0, \epsilon)P^{-1}. \) where \( J(\Lambda_0, \epsilon) \) is defined as follows: for any \( j = 1, \ldots, m \) replace in \( J(\Lambda_0) \) \( \alpha_j \) with \( \alpha_{j, \epsilon}, \) where

\[
\alpha_{j, \epsilon} = \begin{cases} 
\alpha_{j_0} + \epsilon & \text{if } j = j_0, \\
\alpha_j + |\epsilon| & \text{otherwise}.
\end{cases}
\]
Notice that there is $\epsilon_0 > 0$ such that for any $0 < |\epsilon| < \epsilon_0$ an operator $\text{Id} - L_{\Lambda_0, \epsilon} : \mathbb{H} \to \mathbb{H}$ is an isomorphism. Repeating the proof of Lemma 3.4 of [13] we obtain that
\[
\text{deg}_{LS}(\text{Id} - L_{\Lambda_0, \epsilon}, D_\gamma(\mathbb{H}), 0) = \text{deg}_{LS}(\text{Id} - L_{J(\Lambda_0, \epsilon)}, D_\gamma(\mathbb{H}), 0),
\]
where $\text{deg}_{LS}(\cdot, \cdot, \cdot)$ denotes the Leray-Schauder degree. Since $\mu_{-\Delta}(\lambda_k)$ is odd,
\[
\text{deg}_{LS}(\text{Id} - L_{J(\Lambda_0, \epsilon)}, D_\gamma(\mathbb{H}), 0) = -1,
\]
where $-\epsilon_0 < \epsilon' < 0 < \epsilon'' < \epsilon_0$. It is known that a change of the Leray-Schauder degree implies the existence of global bifurcation point, which completes the proof. \qed

**Remark 4.1.1.** If $\Lambda_0 \in \hat{S}_{\text{even}}(m; \mathbb{R})$ then by Theorem 4.1.1 $(0, \Lambda_0) \in \text{BIF}(\nabla_u \Phi)$. Moreover, it can happen that $(0, \Lambda_0) \in \text{BIF}(\nabla_u \Phi) \setminus \text{BRA}(\nabla_u \Phi)$, see [2, 3, 9, 17, 19, 29] for discussion and examples.

From now on we treat $\mathbb{R}^n$ as an orthogonal $\text{SO}(2)$-representation and assume additionally that $\Omega \subset \mathbb{R}^n$ is $\text{SO}(2)$-invariant. Under these assumptions functional (4.2) is $\text{SO}(2)$-invariant and its gradient given by (4.3) is $\text{SO}(2)$-equivariant.

**Theorem 4.1.3.** Under the above assumptions:
\[
\{0\} \times (\hat{S}_{\text{odd}}(m; \mathbb{R}) \cup \hat{S}_{\text{even}}^\text{ess}(m; \mathbb{R})) \subset \mathcal{GLOB}(\nabla_u \Phi).
\]

**Proof.** In view of Theorem 4.1.2 to complete the proof it is enough to show that \( \{0\} \times \hat{S}_{\text{even}}^\text{ess}(m; \mathbb{R}) \subset \mathcal{GLOB}(\nabla_u \Phi) \). Fix $\Lambda_0 \in \hat{S}_{\text{even}}^\text{ess}(m; \mathbb{R})$ and define $\text{SO}(2)$-invariant potential $\Psi : \mathbb{H} \times \mathbb{R} \to \mathbb{R}$ as follows $\Psi(u, \epsilon) = \Phi(u, \Lambda_0 + \epsilon \text{Id}_{\mathbb{H}})$. It is clear that
\[
\text{BIF}_{\text{SO}(2)}(0, \nabla_u \Psi) = \nabla_{\text{SO}(2)} - \text{deg}(\text{Id} - L_{\Lambda_0 + \epsilon \text{Id}_{\mathbb{H}}}, D_\alpha(\mathbb{H})) - \nabla_{\text{SO}(2)} - \text{deg}(\text{Id} - L_{\Lambda_0 - \epsilon \text{Id}_{\mathbb{H}}}, D_\alpha(\mathbb{H})) = \text{BIF}_{\text{SO}(2)}(\Lambda_0),
\]
where $\text{BIF}_{\text{SO}(2)}(\Lambda_0)$ is given by (3.3.1). Since $\Lambda_0 \in \hat{S}_{\text{even}}^\text{ess}(m; \mathbb{R})$, $\text{BIF}_{\text{SO}(2)}(\Lambda_0) \neq \emptyset \in \text{U}(\text{SO}(2))$. The rest of the proof is a direct consequence of Theorem 2.5. \qed

**Remark 4.1.2.** Notice that if $\Lambda_0 \in \hat{S}_{\text{even}}^\text{ess}(m; \mathbb{R}) \setminus \hat{S}_{\text{even}}^\text{ess}(m; \mathbb{R})$ then $\text{BIF}_{\text{SO}(2)}(\Lambda_0) = \emptyset \in \text{U}(\text{SO}(2))$ and by Theorem 4.1.1 $(0, \Lambda_0) \in \text{BIF}(\nabla_u \Phi)$. Moreover, it can happen that $(0, \Lambda_0) \in \text{BIF}(\nabla_u \Phi) \setminus \text{BRA}(\nabla_u \Phi)$.

### 4.2. Bifurcation theorems ($m = 2$).

In this section we study solutions of problem (4.1) with $m = 2$ and $\Lambda = \begin{bmatrix} \lambda & \delta \\ \delta & \gamma \end{bmatrix}$. We identify $\mathbb{R}^3$ with $S(m; \mathbb{R})$ by a map $\psi$ defined by (3.2.6).

Equation (4.1) is equivalent to the following one
\[
\begin{cases}
-\Delta v_1 = (p_2 + p_3)v_1 + p_1 v_2 + \nabla_v \eta(v_1, v_2, \Lambda_p) & \text{in } \Omega, \\
-\Delta v_2 = p_1 v_1 + (-p_2 + p_3)v_2 + \nabla_v \eta(v_1, v_2, \Lambda_p) & \text{in } \Omega, \\
v_1 = v_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

**Theorem 4.2.1.** Under the above assumptions:
\[
\text{BIF}(\nabla_u \Phi) = \{0\} \times S \subset \mathbb{H} \times \mathbb{R}^3.
\]
Proof. To complete the proof it is enough to notice that \( \psi : S \to \widehat{S}(2; \mathbb{R}) \) is a bijection, where \( \psi \) is a map defined by (3.2.6). The rest of the proof is a direct consequence of Theorem 4.1.1. \( \blacksquare \)

**Theorem 4.2.2.** Under the above assumptions:

\[
\{0\} \times S_{\text{odd}} \subset \{0\} \times G\text{LOB}(\nabla u \Phi) \subset \mathbb{H} \times \mathbb{R}^3.
\]

**Proof.** From Lemma 3.2.1 it follows that \( \psi : S_{\text{odd}} \to \widehat{S}_{\text{odd}}(2; \mathbb{R}) \) is a bijection, where \( \psi \) is a map defined by (3.2.6). The rest of the proof is a direct consequence of Theorem 4.1.2. \( \blacksquare \)

From now on we treat \( \mathbb{R}^n \) as an orthogonal \( \text{SO}(2) \)-representation and assume additionally that \( \Omega \subset \mathbb{R}^n \) is \( \text{SO}(2) \)-invariant. Under these assumptions functional (4.2) is \( \text{SO}(2) \)-invariant and its gradient given by (4.3) is \( \text{SO}(2) \)-equivariant.

Put \( \widehat{S} = \{ p \in S \setminus S_{\text{odd}} : \Lambda_p \in \widehat{S}_{\text{ess}}(2; \mathbb{R}) \} \). In other words if \( p \in \widehat{S} \) then \( \mathbb{H}_{(0, \Lambda_p)} \) is an even-dimensional nontrivial \( \text{SO}(2) \)-representation.

**Theorem 4.2.3.** Under the above assumptions:

\[
\{0\} \times (S_{\text{odd}} \cup \widehat{S}) \subset G\text{LOB}(\nabla u \Phi) \subset \mathbb{H} \times \mathbb{R}^3.
\]

**Proof.** If \( p \in S_{\text{odd}} \) then from Theorem 4.2.2 it follows that \( (0, \Lambda_p) \in G\text{LOB}(\nabla u \Phi) \). Fix \( p \in \widehat{S} \) and define an \( \text{SO}(2) \)-invariant functional \( \Psi : \mathbb{H} \times \mathbb{R} \to \mathbb{R} \) as follows \( \Psi(u, \epsilon) = \Phi(u, \Lambda_p + \epsilon \text{Id}_{\mathbb{R}^m}) \). It is clear that

\[
\BLF_{\text{SO}(2)}(0, \nabla u \Psi) = \nabla_{\text{SO}(2)} \sigma(\Lambda_p) - \deg(I - L_{\Lambda_p} \text{Id}_{\mathbb{R}^m})D_\alpha \mathbb{H}) - \nabla_{\text{SO}(2)} \sigma(\Lambda_p) - \deg(I - L_{\Lambda_p} \text{Id}_{\mathbb{R}^m})D_\alpha \mathbb{H}) = \BLF_{\text{SO}(2)}(\Lambda_p),
\]

where \( \BLF_{\text{SO}(2)}(\Lambda_p) \) is given by (3.3.1). Since \( \Lambda_p \in \widehat{S}_{\text{ess}}(2; \mathbb{R}) \), \( \BLF_{\text{SO}(2)}(\Lambda_p) \neq \emptyset \in U(\text{SO}(2)) \). The rest of the proof is a direct consequence of Theorem 2.5. \( \blacksquare \)

5. Examples

It was shown by in [20, 22] that the eigenvalues of the Laplace operator with Dirichlet boundary conditions are simple for most bounded \( C^3 \)-regions. On the other hand it was proved in [22] that this is not true if we restrict our attention to the subset of regions which are invariant under a group of symmetries.

In other to illustrate the abstract results proved in this article we consider three examples.

**Example 5.1.** Let \( \Omega \subset \mathbb{R}^n \) be a \( C^3 \)-region such that \( \mu_\Delta(\lambda_k) = 1 \) for any \( \lambda_k \in \sigma(\Delta; \Omega) \). Under the above assumptions the following equalities hold true:

\[
\begin{align*}
\sigma(\Delta; \Omega) &= \sigma_{\text{odd}}(\Delta; \Omega), \\
\sigma_{\text{SO}(2)}(\Delta; \Omega) &= \sigma_{\text{even}}(\Delta; \Omega) = \emptyset,
\end{align*}
\]

\[
\widehat{S}(m; \mathbb{R}) = \widehat{S}_{\text{odd}}(m; \mathbb{R}), \quad \widehat{S}_{\text{even}}(m; \mathbb{R}) = \widehat{S}_{\text{ess}}(m; \mathbb{R}) = \emptyset.
\]

Applying Theorems 4.1.1 and 4.1.2 we obtain the following:

\[
\{0\} \times \widehat{S}_{\text{odd}}(m; \mathbb{R}) \subset G\text{LOB}(\nabla u \Phi) \subset \BLF(\nabla u \Phi) = \{0\} \times \widehat{S}(m; \mathbb{R}).
\]

Since \( \widehat{S}(m; \mathbb{R}) = \widehat{S}_{\text{odd}}(m; \mathbb{R}) \), \( G\text{LOB}(\nabla u \Phi) = \BLF(\nabla u \Phi) \).
Example 5.2. Let $\Omega = B^2 \subset \mathbb{R}^2$. It is known that for any $\lambda_k \in \sigma(-\Delta; B^2)$ there is $k' \in \mathbb{N} \cup \{0\}$ such that $V_{-\Delta}(\lambda_k) \approx \mathbb{R}[1, k']$. Under the above assumptions the following equalities hold true:

\[
\sigma_{SO(2)}(-\Delta; B^2) = \{\lambda_k \in \sigma(-\Delta; B^2) : \mu_{-\Delta}(\lambda_k) = 2\},
\]
\[
\sigma_{even}(-\Delta; B^2) = \{\lambda_k \in \sigma(-\Delta; B^2) : \mu_{-\Delta}(\lambda_k) = 2\},
\]
\[
\sigma_{odd}(-\Delta; B^2) = \{\lambda_k \in \sigma(-\Delta; B^2) : \mu_{-\Delta}(\lambda_k) = 1\},
\]
\[
\hat{S}_{odd}(m; \mathbb{R}) = \{\lambda \in \hat{S}(m; \mathbb{R}) : \exists \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; \Omega) \text{ s.t. } \mu_{-\Delta}(\lambda_k) = 1\},
\]
\[
\hat{S}_{even}(m; \mathbb{R}) = \{\lambda \in \hat{S}(m; \mathbb{R}) : \forall \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; B^2) \text{ s.t. } \mu_{-\Delta}(\lambda_k) = 2\},
\]
\[
\hat{S}_{ext}(m; \mathbb{R}) = \{\lambda \in \hat{S}(m; \mathbb{R}) : \forall \lambda_k \in \sigma(\Lambda) \cap \sigma(-\Delta; B^2) \text{ s.t. } \mu_{-\Delta}(\lambda_k) = 2\},
\]
Applying Theorems 4.1.1 and 4.1.3 we obtain the following:

\[
\{0\} \times (\hat{S}_{odd}(m; \mathbb{R}) \cup \hat{S}_{ext}(m; \mathbb{R})) \subset \mathcal{GLOB}(\nabla u \Phi) \subset \mathcal{BLF}(\nabla u \Phi) = \{0\} \times \hat{S}(m; \mathbb{R}).
\]

Notice that still it is not clear if

\[
(\hat{S}(m; \mathbb{R}) \setminus (\hat{S}_{odd}(m; \mathbb{R}) \cup \hat{S}_{ext}(m; \mathbb{R}))) \cap \mathcal{BRA}(\nabla u \Phi) \neq \emptyset.
\]

Since $\hat{S}_{ext}(m; \mathbb{R}) \neq \emptyset$, Theorem 4.1.3 is stronger than Theorem 4.1.2.

Example 5.3. Let $\Omega = B^3 \subset \mathbb{R}^3$. It is known that for any $\lambda_k \in \sigma(-\Delta; B^3)$ there is $k' \in \mathbb{N} \cup \{0\}$ such that $V_{-\Delta}(\lambda_k) \approx \mathbb{R}[1, 0] \oplus \mathbb{R}[1, 1] \oplus \ldots \oplus \mathbb{R}[1, k']$. Under the above assumptions the following equalities hold true:

\[
\sigma(-\Delta; B^3) = \sigma_{odd}(-\Delta; B^3) = \sigma_{even}(-\Delta; B^3) = \sigma_{SO(2)}(-\Delta; B^3) = \emptyset,
\]
\[
\hat{S}(m; \mathbb{R}) = \hat{S}_{odd}(m; \mathbb{R}), \hat{S}_{even}(m; \mathbb{R}) = \hat{S}_{ext}(m; \mathbb{R}) = \emptyset.
\]

Applying Theorems 4.1.1 and 4.1.2 we obtain the following:

\[
\{0\} \times \hat{S}_{odd}(m; \mathbb{R}) \subset \mathcal{GLOB}(\nabla u \Phi) \subset \mathcal{BLF}(\nabla u \Phi) = \{0\} \times \hat{S}(m; \mathbb{R}).
\]

Since $\hat{S}(m; \mathbb{R}) = \hat{S}_{odd}(m; \mathbb{R})$, $\mathcal{GLOB}(\nabla u \Phi) = \mathcal{BLF}(\nabla u \Phi)$.

6. Final Remarks

We emphasize that results obtained in this article for system (1.2) are natural generalizations of well-known results concerning single equation (1.1).

The following two questions are at present far from being solved.

Is it true that $\{0\} \times (\hat{S}(m; \mathbb{R}) \setminus \hat{S}_{odd}(m; \mathbb{R})) \cap \mathcal{BRA}(\nabla u \Phi) \neq \emptyset$?

Is it true that $\{0\} \times (\hat{S}(m; \mathbb{R}) \setminus \hat{S}_{odd}(m; \mathbb{R})) \cap \mathcal{GLOB}(\nabla u \Phi) \neq \emptyset$?

If $\Omega$ is SO(2)-symmetric then the above questions can be formulated in the following way.

Is it true that $\{0\} \times (\hat{S}(m; \mathbb{R}) \setminus (\hat{S}_{odd}(m; \mathbb{R}) \cup \hat{S}_{ext}(m; \mathbb{R}))) \cap \mathcal{BRA}(\nabla u \Phi) \neq \emptyset$?

Is it true that $\{0\} \times (\hat{S}(m; \mathbb{R}) \setminus (\hat{S}_{odd}(m; \mathbb{R}) \cup \hat{S}_{ext}(m; \mathbb{R}))) \cap \mathcal{GLOB}(\nabla u \Phi) \neq \emptyset$?
The problem is that the bifurcation indices computed in terms of the Leray-Schauder degree and the degree for SO(2)-equivariant gradient maps at \( \Lambda \in \hat{S}(m; \mathbb{R}) \setminus \hat{S}_{odd}(m; \mathbb{R}) \) and \( \Lambda \in \hat{S}(m; \mathbb{R}) \setminus (\hat{S}_{odd}(m; \mathbb{R}) \cup \hat{S}_{even}(m; \mathbb{R})) \), respectively, are trivial. On the other hand, change of the Conley index along the path of trivial solutions does not guarantee the existence of branching points.

**References**


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