EXISTENCE AND CONTINUATION OF SOLUTIONS
FOR A NONLINEAR NEUMANN PROBLEM

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ABSTRACT. In this article we study the existence and continuation of nonconstant solutions for a nonlinear Neumann problem. We apply the Leray-Schauder degree and the degree for $SO(2)$-equivariant gradient operators defined by the second author in [13].

1. INTRODUCTION

Consider the following nonlinear Neumann problem

\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with $C^1-$boundary and $f \in C^1(\mathbb{R}, \mathbb{R})$.

The existence and multiplicity of weak solutions of problem (1.1) has been studied by many authors, see for example [6, 8, 9, 16] and references therein.

Usually weak solutions of system (1.1) are considered as critical orbits of a functional $\Phi \in C^2(H^1(\Omega), \mathbb{R})$. The authors apply tools of the critical point theory, like Morse theory, Conley index technique and mountain pass theorem, to obtain results.

The first goal of this article is to prove the sufficient conditions for the existence of solutions of problem (1.1).

Let $\sigma(-\Delta; \Omega) = \{0 = \lambda_1 < \lambda_2 < \ldots\}$ denote the set of eigenvalues of the following eigenvalue problem

\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and let $V_{-\Delta}(\lambda_i)$ be the eigenspace of the Laplace operator $-\Delta$ corresponding to the eigenvalue $\lambda_i \in \sigma(-\Delta; \Omega)$.

We assume that $f$ is asymptotically linear i.e. $f(x) = f'(\infty)x + o(|x|)$, as $|x| \to \infty$ and that $Z = f^{-1}(0)$ is finite.
In our theorems we put assumptions on $f'(z)$, where $z \in Z \cup \{\infty\}$. We emphasize that we also treat problems with resonance at constant solutions and at infinity i.e. it can happen that $f'(z) \in \sigma(-\Delta; \Omega)$ for some $z \in Z \cup \{\infty\}$.

Since the gradient $\nabla \Phi \in C^1(H^1(\Omega), H^1(\Omega))$ is of the form compact perturbation of the identity, we apply the Leray-Schauder degree and the degree for $SO(2)$-equivariant gradient maps to the study of critical points (critical $SO(2)$-orbits) of the functional $\Phi$.

The second goal of this article is to prove the sufficient conditions for continuation of solutions of the following

$$
\begin{cases}
-\Delta u &= f(u, \lambda) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega,
\end{cases}
$$

(1.3)

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with $C^1$-boundary and $f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

It is worth in pointing out that application of classical invariants like the Conley index technique and the Morse theory does not ensure the existence of closed connected sets of critical points of variational problems, see [2, 3, 7, 10, 15] for examples and discussion. In other words we can not apply these invariants in order to prove continuation of solutions of problem (1.3).

That is why again we use the Leray-Schauder degree and the degree for $SO(2)$-equivariant gradient maps to the study of continuation of critical points (critical $SO(2)$-orbits) of the functional $\Phi \in C^2(H^1(\Omega) \times \mathbb{R}, \mathbb{R})$. The choice of the Leray-Schauder degree and the degree for $SO(2)$-equivariant gradient maps seems to be the best adapted to our theory.

After this introduction our article is organized as follows.

Since the degree for $SO(2)$-equivariant gradient maps is not widely known, in Section 2 we have summarized without proofs the relevant material on this invariant, thus making our exposition as self-contained as possible.

In Section 3 we have studied problem (1.2). In Lemma 3.1 we have derived a formula for the Leray-Schauder degree of the gradient $\nabla \Psi \in C^1(H^1(\Omega) \times \mathbb{R}, H^1(\Omega))$ of a functional $\Psi \in C^2(H^1(\Omega) \times \mathbb{R}, \mathbb{R})$ associated with problem (1.2). Suppose now that $\mathbb{R}^n$ is an orthogonal $SO(2)$-representation and that $\Omega \subset \mathbb{R}^n$ is $SO(2)$-invariant. Under these assumptions $H^1(\Omega)$ is an orthogonal $SO(2)$-representation, the functional $\Psi$ is $SO(2)$-invariant and its gradient $\nabla \Psi$ is $SO(2)$-equivariant. In Lemma 3.2 we have proved a formula for the degree for $SO(2)$-equivariant gradient maps of $\nabla \Psi$.

In Section 4 our main results are stated and proved. Subsection 4.1 is devoted to the study of the existence of nonconstant solutions of problem (1.1). In Theorems 4.1.1-4.1.4 we consider non-degenerate case i.e. we assume that $f'(z) \notin \sigma(-\Delta; \Omega)$ for every $z \in Z \cup \{\infty\}$. These theorems ensure the existence of at least one nonconstant solution of problem (1.1). Notice that in Theorems 4.1.2-4.1.4 we have assumed that domain $\Omega$ is $SO(2)$-invariant.

We emphasize that in the proofs of Theorems 4.1.2-4.1.4 the degree for $SO(2)$-equivariant gradient maps can not be replaced with the Leray-Schauder degree, see Remark 4.1.3. Additionally, in Theorem 4.1.5 we have proved the existence of at least one nonconstant solution of problem (1.1) in a degenerate case.
In Subsection 4.2 we have studied continuation of nonconstant solutions of problem (1.3).

In Section 5 we consider three real problems in order to illustrate the main results of this paper. To illustrate results proved in this article we consider problem (1.1) with \( \Omega = B^2 \) and \( \Omega = (0,1) \times B^2 \).

2. Preliminaries

In this section, for the convenience of the reader, we remind the main properties of the degree for \( SO(2) \)-equivariant gradient maps defined in [13]. This degree will be denoted briefly by \( \nabla_{SO(2)} \)-deg.

Denote by \( Y(SO(2)) \) the set of closed subgroups of the group \( SO(2) \) i.e. \( Y(SO(2)) = \{ SO(2), \mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_k, \ldots \} \).

Put \( U(SO(2)) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z} \) and define actions

\[ +, \star : U(SO(2)) \times U(SO(2)) \to U(SO(2)), \]
\[ \cdot : \mathbb{Z} \times U(SO(2)) \to U(SO(2)), \]

as follows

\[ \alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k, \ldots), \]
\[ \alpha \star \beta = (\alpha_0 \cdot \beta_0, \alpha_0 \cdot \beta_1 + \beta_0 \cdot \alpha_1, \ldots, \alpha_0 \cdot \beta_k + \beta_0 \cdot \alpha_k, \ldots), \]
\[ \gamma \cdot \alpha = (\gamma \cdot \alpha_0, \gamma \cdot \alpha_1, \ldots, \gamma \cdot \alpha_k, \ldots), \]

where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots), \beta = (\beta_0, \beta_1, \ldots, \beta_k, \ldots) \in U(SO(2)) \) and \( \gamma \in \mathbb{Z} \). It is easy to check that \( U(SO(2)) \) is a commutative ring with the unit \( \mathbb{I} = (1,0,\ldots) \in (SO(2)) \) and the trivial element \( \Theta = (0,0,\ldots) \in U(SO(2)) \). Ring \( (U(SO(2)), +, \star) \) is called the tom Dieck ring of the group \( SO(2) \). For a definition of the tom Dieck ring \( U(G) \), where \( G \) is any compact Lie group, we refer the reader to [4].

If \( \delta_1, \ldots, \delta_q \in U(SO(2)) \), then we write \( \prod_{j=1}^{q} \delta_j \) for \( \delta_1 \star \ldots \star \delta_q \). Moreover, it is understood that \( \prod_{j \in \emptyset} \delta_j = \mathbb{I} \in U(SO(2)) \).

Let \( \mathcal{V} \) be a real, finite-dimensional and orthogonal \( SO(2) \)-representation. If \( v \in \mathcal{V} \), then the subgroup \( SO(2)_v = \{ g \in SO(2) : g \cdot v = v \} \) is said to be the isotropy group of \( v \in \mathcal{V} \).

Let \( \Omega \subset \mathcal{V} \) be an open, bounded and \( SO(2) \)-invariant subset and let \( H \in Y(SO(2)) \).

Then we define

- \( \Omega^H = \{ v \in \Omega : H \subset SO(2)_v \} = \{ v \in \Omega : gv = v \ \forall \ g \in H \}, \)
- \( \Omega_H = \{ v \in \Omega : H = SO(2)_v \} \).

Fix \( k \in \mathbb{N} \) and set

- \( C^k_{SO(2)}(\mathcal{V}, \mathbb{R}) = \{ f \in C^k(\mathcal{V}, \mathbb{R}) : f \text{ is } SO(2)\text{-invariant} \}, \)
- \( C_{SO(2)}^{-k-1}(\mathcal{V}, \mathcal{V}) = \{ f \in C^{-k-1}(\mathcal{V}, \mathcal{V}) : f \text{ is } SO(2)\text{-equivariant} \}. \)
Let \( f \in C^1_{SO(2)}(V, \mathbb{R}) \). Since \( V \) is an orthogonal \( SO(2) \)-representation, the gradient \( \nabla f \in C^0_{SO(2)}(V, V) \). If \( H \in \mathcal{Y}(SO(2)) \) is a closed subgroup, then \( V^H \) is a finite-dimensional \( SO(2) \)-representation and \( (\nabla f)^H = \nabla (f_{V^H}) : V^H \to V^H \) is well-defined \( SO(2) \)-equivariant gradient map. Choose an open, bounded and \( SO(2) \)-invariant subset \( \Omega \subset V \) such that \( (\nabla f)^{-1}(0) \cap \partial \Omega = \emptyset \). Under these assumptions we have defined in [13] the degree for \( SO(2) \)-equivariant gradient maps \( \nabla_{SO(2)} - \deg(\nabla f, \Omega) \in U(SO(2)) \) with coordinates

\[
\nabla_{SO(2)} - \deg(\nabla f, \Omega) = \\
= (\nabla_{SO(2)} - \deg_{SO(2)}(\nabla f, \Omega), \nabla_{SO(2)} - \deg_{Z_1}(\nabla f, \Omega), \ldots, \nabla_{SO(2)} - \deg_{Z_k}(\nabla f, \Omega), \ldots).
\]

**Remark 2.1.** To define the degree for \( SO(2) \)-equivariant gradient maps of \( \nabla f_0 \) we choose (in a homotopy class of the \( SO(2) \)-equivariant gradient map \( \nabla f_0 \)) a sufficiently good \( SO(2) \)-equivariant gradient map \( \nabla f_1 \) and define this degree for \( \nabla f_1 \). The definition does not depend on the choice of the map \( \nabla f_1 \). Roughly speaking the main steps of the definition of the degree for \( SO(2) \)-equivariant gradient maps of \( \nabla f_0 : (\mathcal{E}(\Omega), \partial \Omega) \to (V, V \setminus \{0\}) \) are the following:

**Step 1.** There is a potential \( f \in C^1_{SO(2)}(V \times [0,1], \mathbb{R}) \) such that

(a1) \( (\nabla_v f)^{-1}(0) \cap (\partial \Omega \times [0,1]) = \emptyset \),

(a2) \( \nabla_v f(\cdot, 0) = \nabla f_0(\cdot) \),

(a3) \( \nabla_v f_1 \in C^1_{SO(2)}(V, V) \), where we abbreviate \( \nabla_v f(\cdot, 1) \) to \( \nabla_v f_1 \),

(a4) \( (\nabla_v f_1)^{-1}(0) \cap \Omega^{SO(2)} = \{ v_1, \ldots, v_p \} \) and

(i) \( \det \nabla_{vv}^2 f_1(v_j) \neq 0 \), for all \( j = 1, \ldots, p \),

(ii) \( \nabla_{vv}^2 f_1(v_j) = \begin{bmatrix} \nabla_{vv}^2 (f_1^{SO(2)})(v_j) & 0 \\ 0 & Id \end{bmatrix} : V^{SO(2)} \oplus (V^{SO(2)})^\perp \to V^{SO(2)} \oplus (V^{SO(2)})^\perp \),

for all \( j = 1, \ldots, p \),

(a5) \( (\nabla_v f_1)^{-1}(0) \cap (\Omega \setminus \Omega^{SO(2)}) = \{ SO(2)w_1, \ldots, SO(2)w_q \} \) and

(i) \( \dim \ker \nabla_{vv} f_1(w_j) = 1 \), for all \( j = 1, \ldots, q \),

(ii) \( \nabla_{vv}^2 f_1(w_j) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_j & 0 \\ 0 & 0 & Id \end{bmatrix} : T_{w_j}(SO(2)w_j) \oplus T_{w_j}(SO(2)w_j) \oplus (T_{w_j}(V_{SO(2)w_j}))^\perp \to T_{w_j}(V_{SO(2)w_j}) \oplus (T_{w_j}(V_{SO(2)w_j}))^\perp \),

for all \( j = 1, \ldots, q \).

**Step 2.** The first coordinate of the degree for \( SO(2) \)-equivariant gradient maps is defined by \( \nabla_{SO(2)} - \deg_{SO(2)}(\nabla f_0, \Omega) = \sum_{j=1}^p \sign \det \nabla_{vv}^2 (f_1^{SO(2)})(v_j) \). In other words since
\[ \nabla(f_1^{SO(2)}) = (\nabla f_1)^{SO(2)} \], we obtain
\[ \nabla_{SO(2)} - \deg_{SO(2)}(\nabla f_0, \Omega) = \deg_B((\nabla f_1)^{SO(2)}, \Omega^{SO(2)}, 0), \]
where \( \deg_B \) denotes the Brouwer degree.

Step 3. Fix \( k \in \mathbb{N} \) and define
\[ \nabla_{SO(2)} - \deg_{\mathbb{Z}_k}(\nabla f_0, \Omega) = \sum_{j \in \{1, \ldots, q\} : SO(2)_{\mathbb{Z}_j} = \mathbb{Z}_k} \text{sign det } Q_j, \]
Notice that since
\[ \deg_B((\nabla f_1)^{SO(2)}, \Omega^{SO(2)}, 0) = \deg_B(\nabla f_1, \Omega, 0) \] and \( \deg_B(\nabla f_1, \Omega, 0) = \deg_B(\nabla f_0, \Omega, 0) \) (see [12]), directly by the Step 2. we obtain \( \nabla_{SO(2)} - \deg_{SO(2)}(\nabla f_0, \Omega) = \deg_B(\nabla f_0, \Omega, 0) \). Moreover, immediately from the Step 3. we obtain that if \( k \in \mathbb{N} \) and \( SO(2)_v \neq \mathbb{Z}_k \) for every \( v \in \Omega \), then \( \nabla_{SO(2)} - \deg_{\mathbb{Z}_k}(\nabla f_0, \Omega) = 0 \).

For \( \gamma > 0 \) and \( v_0 \in \nabla^{SO(2)} \) we put \( B_\gamma(\nabla, v_0) = \{ v \in \nabla : |v - v_0| < \gamma \} \) and \( D_\gamma(\nabla, v_0) = \{ v \in \nabla : |v - v_0| \leq \gamma \} \). For simplicity of notation we put \( B_\gamma(\nabla) = B_\gamma(\nabla, 0) \) and \( D_\gamma(\nabla) = D_\gamma(\nabla, 0) \).

In the following theorem we formulate the main properties of the degree for \( SO(2) \)-equivariant gradient maps.

**Theorem 2.1** ([13]). Under the above assumptions the degree for \( SO(2) \)-equivariant gradient maps has the following properties

1. if \( \nabla_{SO(2)} - \deg(\nabla f, \Omega) \neq \emptyset \), then \( (\nabla f)^{-1}(0) \cap \Omega \neq \emptyset \),
2. if \( \nabla_{SO(2)} - \deg_H(\nabla f, \Omega) \neq 0 \), then \( (\nabla f)^{-1}(0) \cap \Omega^H \neq \emptyset \),
3. if \( \Omega = \Omega_0 \cup \Omega_1 \) and \( \Omega_0 \cap \Omega_1 = \emptyset \), then
   \[ \nabla_{SO(2)} - \deg(\nabla f, \Omega) = \nabla_{SO(2)} - \deg(\nabla f, \Omega_0) + \nabla_{SO(2)} - \deg(\nabla f, \Omega_1), \]
4. if \( \Omega_0 \subset \Omega \) is an open \( SO(2) \)-invariant subset and \( (\nabla f)^{-1}(0) \cap \Omega \subset \Omega_0 \), then
   \[ \nabla_{SO(2)} - \deg(\nabla f, \Omega) = \nabla_{SO(2)} - \deg(\nabla f, \Omega_0), \]
5. if \( f \in C^1_{SO(2)}(\nabla \times [0, 1], \mathbb{R}) \) is such that \( (\nabla_v f)^{-1}(0) \cap (\partial \Omega \times [0, 1]) = \emptyset \), then
   \[ \nabla_{SO(2)} - \deg(\nabla f_0, \Omega) = \nabla_{SO(2)} - \deg(\nabla f_1, \Omega), \]
6. if \( W \) is an orthogonal \( SO(2) \)-representation, then
   \[ \nabla_{SO(2)} - \deg((\nabla f, Id, \Omega \times B_\gamma(W)) = \nabla_{SO(2)} - \deg(\nabla f, \Omega), \]
7. if \( f \in C^2_{SO(2)}(\nabla, \mathbb{R}) \) is such that \( \nabla f(0) = 0 \) and \( \nabla^2 f(0) \) is an \( SO(2) \)-equivariant self-adjoint isomorphism, then there is \( \gamma > 0 \) such that
   \[ \nabla_{SO(2)} - \deg(\nabla f, B_\gamma(\nabla)) = \nabla_{SO(2)} - \deg(\nabla^2 f(0), B_\gamma(\nabla)). \]

**Remark 2.2.** Directly from the definition of the degree for \( SO(2) \)-equivariant gradient maps (see [13]) it follows that

1. if \( H \in \Upsilon(SO(2)) \) is a closed subgroup and \( SO(2)_v \neq H \), for every \( v \in \Omega \), then
   \[ \nabla_{SO(2)} - \deg_H(\nabla f, \Omega) = 0. \]
Below we formulate product formula for the degree for $SO(2)$-equivariant gradient maps.

**Theorem 2.2** ([14]). Let $\Omega_i \subset V_i$ be an open, bounded and $SO(2)$-invariant subset of a finite-dimensional, orthogonal $SO(2)$-representation $V_i$ of the group $SO(2)$, for $i = 1, 2$. Let $f_i \in C^1_{SO(2)}(V_i, \mathbb{R})$ be such that $(\nabla f_i)^{-1}(0) \cap \partial \Omega_i = \emptyset$, for $i = 1, 2$. Then

$$\nabla_{SO(2)} - \text{deg}((\nabla f_1, \nabla f_2), \Omega_1 \times \Omega_2) = \nabla_{SO(2)} - \text{deg}(\nabla f_1, \Omega_1) + \nabla_{SO(2)} - \text{deg}(\nabla f_2, \Omega_2).$$

For $k \in \mathbb{N}$ define a map $\rho^k : SO(2) \to GL(2, \mathbb{R})$ as follows

$$\rho^k \left( \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \right) = \begin{bmatrix} \cos(k \cdot \theta) & -\sin(k \cdot \theta) \\ \sin(k \cdot \theta) & \cos(k \cdot \theta) \end{bmatrix} \quad 0 \leq \theta < 2 \cdot \pi.$$ 

For $j, k \in \mathbb{N}$ we denote by $\mathbb{R}[j, k]$ the direct sum of $j$ copies of $(\mathbb{R}^2, \rho^k)$, we also denote by $\mathbb{R}[j, 0]$ the trivial $j$-dimensional $SO(2)$-representation. We say that two $SO(2)$-representations $V$ and $W$ are equivalent if there exists an $SO(2)$-equivariant, linear isomorphism $T : V \rightarrow W$. The following classic result gives a complete classification (up to equivalence) of finite-dimensional $SO(2)$-representations (see [1]).

**Theorem 2.3** ([1]). If $V$ is a finite-dimensional $SO(2)$-representation, then there exist finite sequences $\{j_i\}, \{k_i\}$ satisfying:

\begin{equation}
(*) \quad k_i \in \{0\} \cup \mathbb{N}, \quad j_i \in \mathbb{N}, \quad 1 \leq i \leq r, \quad k_1 \leq k_2 \leq \cdots \leq k_r
\end{equation}

such that $V$ is equivalent to $\bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]$. Moreover, the equivalence class of $V$ ($V \approx \bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]$) is uniquely determined by $\{k_i\}, \{j_i\}$ satisfying $(*)$.

Notice that if $V \approx \bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]$ and $k_1 = 0$, then $V^{SO(2)} \approx \mathbb{R}[j_1, 0]$. An $SO(2)$-representation $V$ is called nontrivial if $V^{SO(2)} \neq V$. Suppose that $j' \in \mathbb{N}, k' \in \mathbb{N} \cup \{0\}$ and $V \approx \bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]$. It is understood that if $\mathbb{R}[1, k'] \not\subset V$, then $k' \neq k_i$ for $i = 1, \ldots, r$.

Moreover, if $k' \in \mathbb{N}$, then $V_{Z_{\mathbb{R}}} = \emptyset$ is equivalent to $k' \neq \gcd(k_{i_1}, \ldots, k_{i_s})$ for every $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$.

We will denote by $m^-(L)$ the Morse index of a symmetric matrix $L$.

To apply successfully any degree theory we need computational formulas for this invariant. Below we show how to compute degree for $SO(2)$-equivariant gradient maps of a linear, self-adjoint, $SO(2)$-equivariant isomorphism.

**Lemma 2.1** ([13]). If $V \approx \mathbb{R}[j_0, 0] \oplus \mathbb{R}[j_1, k_1] \oplus \cdots \oplus \mathbb{R}[j_r, k_r]$, $L : V \rightarrow V$ is a self-adjoint, $SO(2)$-equivariant, linear isomorphism and $\gamma > 0$, then

\begin{enumerate}
\item $L = \text{diag}(L_0, L_1, \ldots, L_r)$,
\end{enumerate}
Theorem 2.5. Let $\Phi \in C^1_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ be such that $\nabla_a \Phi(u, \lambda) = u - \nabla_a \eta(u, \lambda)$, where $\nabla \eta : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ is an $SO(2)$-equivariant compact operator. Fix an open, bounded and $SO(2)$-invariant subset $U \subset \mathbb{H}$ and $\lambda_0 \in \mathbb{R}$ such that

$$\nabla_{SO(2)} - \text{deg}(I_d - L, B_\gamma(\mathbb{H})) = \prod_{\lambda_i > 1} \nabla_{SO(2)} - \text{deg}(-I_d, B_\gamma(V_L(\lambda_i))) \in U(SO(2)).$$

It is understood that if $\sigma(L) \cap [1, +\infty) = \emptyset$, then

$$\nabla_{SO(2)} - \text{deg}(I_d - L, B_\gamma(\mathbb{H})) = 1 \in U(SO(2)).$$

Below we formulate the continuation theorem for $SO(2)$-equivariant gradient operators of the form compact perturbation of the identity. In other words we study continuation of critical orbits of $SO(2)$-invariant $C^1$-functionals. The proof of this theorem is standard, but in the proof we have to replace the Leray-Schauder degree with the degree for $SO(2)$-equivariant gradient operators.

Theorem 2.4. Under the above assumptions if $1 \notin \sigma(L)$, then

$$\nabla_{SO(2)} - \text{deg}(I_d - L, B_\gamma(\mathbb{H})) = \prod_{\lambda_i > 1} \nabla_{SO(2)} - \text{deg}(-I_d, B_\gamma(V_L(\lambda_i))) \in U(SO(2)).$$

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle_\mathbb{H})$ be an infinite-dimensional, separable Hilbert space which is an orthogonal $SO(2)$-representation and let $C^1_{SO(2)}(\mathbb{H}, \mathbb{R})$ denote the set of $SO(2)$-invariant $C^1$-functionals. Fix $\Phi \in C^1_{SO(2)}(\mathbb{H}, \mathbb{R})$ such that

$$\nabla \Phi(u) = u - \nabla \eta(u),$$

where $\nabla \eta : \mathbb{H} \to \mathbb{H}$ is an $SO(2)$-equivariant compact operator. Let $U \subset \mathbb{H}$ be an open, bounded and $SO(2)$-invariant set such that $(\nabla \Phi)^{-1}(0) \cap \partial U = \emptyset$. In this situation $\nabla_{SO(2)} - \text{deg}(I_d - \nabla \eta, U) \in U(SO(2))$ is well-defined, see [13] for details and properties of this degree.

Let $L : \mathbb{H} \to \mathbb{H}$ be a linear, bounded, self-adjoint, $SO(2)$-equivariant operator with spectrum $\sigma(L) = \{\lambda_i\}$. By $V_L(\lambda_i)$ we will denote eigenspace of $L$ corresponding to the eigenvalue $\lambda_i$ and we put $\mu_L(\lambda_i) = \dim V_L(\lambda_i)$. In other words $\mu_L(\lambda_i)$ is the multiplicity of the eigenvalue $\lambda_i$. Since operator $L$ is linear, bounded, self-adjoint, and $SO(2)$-equivariant, $V_L(\lambda_i)$ is a finite-dimensional, orthogonal $SO(2)$-representation.

For $\gamma > 0$ and $v_0 \in \mathbb{H}^{SO(2)}$ put $B_\gamma(\mathbb{H}, v_0) = \{v \in \mathbb{H} : |v - v_0| < \gamma\}$. For simplicity of notation $B_\gamma(\mathbb{H})$ stands for $B_\gamma(\mathbb{H}, 0)$.

Combining Theorem 4.5 in [13] with Theorem 2.2 we obtain the following theorem.

Theorem 2.4. Under the above assumptions if $1 \notin \sigma(L)$, then

$$\nabla_{SO(2)} - \text{deg}(I_d - L, B_\gamma(\mathbb{H})) = \prod_{\lambda_i > 1} \nabla_{SO(2)} - \text{deg}(-I_d, B_\gamma(V_L(\lambda_i))) \in U(SO(2)).$$

It is understood that if $\sigma(L) \cap [1, +\infty) = \emptyset$, then

$$\nabla_{SO(2)} - \text{deg}(I_d - L, B_\gamma(\mathbb{H})) = 1 \in U(SO(2)).$$

\[\begin{align*}
\nabla_{SO(2)} - \text{deg}(I_d - L, B_\gamma(\mathbb{H})) &= \prod_{\lambda_i > 1} \nabla_{SO(2)} - \text{deg}(-I_d, B_\gamma(V_L(\lambda_i))) \\
&\in U(SO(2)).
\end{align*}\]
Then there exist continua (closed connected sets) $C^\pm \subset \mathbb{H} \times \mathbb{R}$, with

$$C^- \subset (\mathbb{H} \times (-\infty, \lambda_0]) \cap (\nabla u \Phi(\cdot, \lambda_0))^{-1}(0),$$

$$C^+ \subset (\mathbb{H} \times [\lambda_0, +\infty)) \cap (\nabla u \Phi(\cdot, \lambda_0))^{-1}(0),$$

and for both $C = C^\pm$ the following statements are valid

1. $C \cap (U \times \{\lambda_0\}) \neq \emptyset$,
2. either $C$ is unbounded or else $C \cap ((\mathbb{H} \setminus \text{cl}(U)) \times \{\lambda_0\}) \neq \emptyset$.

3. Linear equation

Throughout this section we assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open set with $C^1-$ boundary. Consider the following eigenvalue problem

$$\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}$$

(3.1)

Denote by $\sigma(-\Delta; \Omega) := \{0 = \lambda_1 < \lambda_2 < \ldots \}$ the distinct eigenvalues of (3.1). Let $V_{-\Delta}(\lambda_i)$ be the eigenspace of $-\Delta$ corresponding to the eigenvalue $\lambda_i \in \sigma(-\Delta; \Omega)$. Additionally define

$$\nu(\lambda) = \begin{cases}
\sum_{\lambda_i < \lambda} \dim V_{-\Delta}(\lambda_i) & \text{if } \lambda > 0, \\
0 & \text{if } \lambda \leq 0.
\end{cases}$$

Solutions of problem (3.1) are in one to one correspondence with critical points of functional $\Psi : H^1(\Omega) \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Psi(u, \lambda) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \lambda u^2 \, dx.$$  

Computing the gradient $\nabla_u \Psi : H^1(\Omega) \times \mathbb{R} \to H^1(\Omega)$ we obtain

$$\langle \nabla_u \Psi(u, \lambda), v \rangle_{H^1(\Omega)} = \int_{\Omega} \nabla u \nabla v - \lambda uv \, dx =$$

$$= \int_{\Omega} \nabla u \nabla v + uv - uv - \lambda uv \, dx = \langle u, v \rangle_{H^1(\Omega)} - (\lambda + 1) \int_{\Omega} uv \, dx.$$  

According to the Riesz theorem there exists linear bounded operator $K : H^1(\Omega) \to H^1(\Omega)$ given by formula $\langle Ku, v \rangle_{H^1(\Omega)} = \int_{\Omega} uv \, dx$. By definition $K$ is self adjoint and by the imbedding theorems it is compact. Hence, $\nabla_u \Psi(u, \lambda) = u - (\lambda + 1)Ku$.

Fix $\lambda_i \in \sigma(-\Delta; \Omega)$ and $u_i \in V_{-\Delta}(\lambda_i)$. Thus $\nabla_u \Psi(u_i, \lambda_i) = 0$ and consequently

$$\nabla_u \Psi(u_i, \lambda) = u_i - (\lambda + 1)Ku_i = u_i - \frac{\lambda + 1}{\lambda_i + 1}u_i = \frac{\lambda_i - \lambda}{\lambda_i + 1}u_i.$$  

(3.2)
By the spectral theorem for compact, self-adjoint operators $H^1(\Omega) = \bigoplus_{i=1}^{\infty} \mathbb{V}_{-\Delta}(\lambda_i)$. Moreover, for every $u \in H^1(\Omega)$ there exists a unique representation $u = \sum_{i=1}^{\infty} u_i$ such that $u_i \in \mathbb{V}_{-\Delta}(\lambda_i)$ for $i \in \mathbb{N} \cup \{0\}$. Hence by (3.2) we obtain

$$\nabla_u \Psi(u, \lambda) = u - (1 + \lambda)Ku = \sum_{i=0}^{\infty} \left( \frac{\lambda_i - \lambda}{\lambda_i + 1} \right) u_i. \quad (3.3)$$

Since $\nabla_u \Psi(\cdot, \lambda)$ is a family of operators of the form compact perturbation of the identity, one can apply the Leray-Schauder $\deg_{LS}$ degree to $\nabla_u \Psi(\cdot, \lambda)$.

The standard proof of the following lemma is omitted.

**Lemma 3.1.** Fix $\lambda \not\in \sigma(-\Delta; \Omega)$ and $\gamma > 0$. Then

$$\deg_{LS}(\nabla_u \Psi(\cdot, \lambda), B_\gamma(H^1(\Omega)), 0) = (-1)^{\nu(\lambda)}.$$

**Remark 3.1.** If $\lambda \in (0, +\infty) \setminus \sigma(-\Delta; \Omega)$, then

$$\deg_{LS}(\nabla_u \Psi(\cdot, \lambda), B_\gamma(H^1(\Omega)), 0) = \prod_{\lambda_i < \lambda} \deg_{LS}(-Id, B_\gamma(\mathbb{V}_{-\Delta}(\lambda_i)), 0) \in \{\pm 1\}.$$

If $\lambda < 0$, then it is understood that $\deg_{LS}(\nabla_u \Psi(\cdot, \lambda), B_\gamma(H^1(\Omega)), 0) = 1$.

**Remark 3.2.** Consider $\mathbb{V} = \mathbb{R}^n$ as an orthogonal $SO(2)$-representation and let $\Omega \subset \mathbb{V}$ be $SO(2)$-invariant. Then $H^1(\Omega)$ is an orthogonal $SO(2)$-representation with an action given by $(gu)(x) = u(gx)$. For every $\lambda_i \in \sigma(-\Delta; \Omega)$, $\mathbb{V}_{-\Delta}(\lambda_i)$ is an orthogonal finite-dimensional $SO(2)$-representation. Moreover, since $\Psi$ is $SO(2)$-invariant, $\nabla_u \Psi$ is $SO(2)$-equivariant.

**Lemma 3.2.** Assume that $\Omega \subset \mathbb{V}$ is $SO(2)$-invariant. Fix $\lambda \not\in \sigma(-\Delta; \Omega)$ and $\gamma > 0$. Then

$$\nabla_{SO(2)}\deg(\nabla_u \Psi(\cdot, \lambda), B_\gamma(H^1(\Omega))) = \prod_{\lambda_i < \lambda} \nabla_{SO(2)}\deg(-Id, B_\gamma(\mathbb{V}_{-\Delta}(\lambda_i))) \in \mathbb{U}(SO(2)).$$

It is understood that if $\lambda < 0$, then

$$\nabla_{SO(2)}\deg(\nabla_u \Psi(\cdot, \lambda), B_\gamma(H^1(\Omega))) = \mathbb{I} \in \mathbb{U}(SO(2)).$$

**Proof.** From (3.3) it follows that $\sigma((1 + \lambda)K) = \{ \frac{\lambda_i + 1}{\lambda_i + 1} : \lambda_i \in \sigma(-\Delta; \Omega) \}$. By assumption $1 \not\in \sigma((1 + \lambda)K)$. Applying Theorem 2.4 we obtain

$$\nabla_{SO(2)}\deg(\nabla_u \Psi(\cdot, \lambda), B_\gamma(H^1(\Omega))) = \nabla_{SO(2)}\deg(-Id - (\lambda + 1)K, B_\gamma(H^1(\Omega))) = \prod_{\frac{\lambda_i + 1}{\lambda_i + 1} > 1} \nabla_{SO(2)}\deg(-Id, B_\gamma(\mathbb{V}_{-\Delta}(\lambda_i))),$$

which completes the proof. □
4. Results

In this section we formulate and prove the main results of this article. In the first subsection we formulate the sufficient conditions for the existence of nonconstant solutions of the following equation

\[
\begin{aligned}
-\Delta u &= f(u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

In the second subsection we study continuation of solutions of the following family of equations

\[
\begin{aligned}
-\Delta u &= f(u, \lambda) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

In the proofs of theorems of this section as the topological invariants we use the Leray-Schauder degree and the degree for SO(2)-equivariant gradient maps.

4.1. Existence of nonconstant solutions. In this section we study weak solutions of the following equation

\[
\begin{aligned}
-\Delta u &= f(u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(4.1.1)

where \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^1 \)-boundary and \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfy the following assumption

\[(A.1) \quad |f'(x)| \leq a + b|x|^{2^*-2} \quad \text{for some } a, b > 0, \text{ where } 2^* = \frac{2n}{n-2}, \text{ for } n \geq 3 \text{ and } 2^* = \infty \text{ for } n = 1, 2.\]

Set \( F : \mathbb{R} \to \mathbb{R} \) a primitive of \( f \) i.e. \( F(t) = \int_0^t f(s)ds \). Weak solutions of equation (4.1.1) are in one to one correspondence with critical points of a functional \( \Phi \in C^2(H^1(\Omega), \mathbb{R}) \) defined by \( \Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega F(u)dx \).

**Remark 4.1.1.** Constant function \( z_0 \in H^1(\Omega) \) is a critical point of \( \Phi \) iff \( z_0 \in Z = f^{-1}(0) \).

Fix \( z_0 \in Z \). Since \( \nabla^2 \Phi(z_0) = Id - (1 + f'(z_0))K \) and (3.3) it follows that \( \nabla^2 \Phi(z_0) : H^1(\Omega) \to H^1(\Omega) \) is an isomorphism iff \( f'(z_0) \not\in \sigma(-\Delta; \Omega) \).

Let us put the following additional assumption

\[(A.2) \quad \text{there exists limit } f'(\infty) = \lim_{|x| \to \infty} \frac{f(x)}{x}.\]

Notice that \( \nabla \Phi(u) = \nabla^2 \Phi(\infty)u + o(|u|_{H^1(\Omega)}) = u - (1 + f'(\infty))Ku + o(|u|_{H^1(\Omega)}) \) as \( |u|_{H^1(\Omega)} \to \infty \).

We treat \( \infty \) as a critical point of \( \Phi \). We say that \( \infty \) is an isolated critical point of \( \Phi \) if \( (\nabla \Phi)^{-1}(0) \) is bounded. Assume that all the elements of \( Z \cup \{ \infty \} \) are isolated critical points of \( \Phi \). From now on \( \gamma_z \) denotes a positive real number such that:

(i) if \( z \in Z \), then \( (\nabla \Phi)^{-1}(0) \cap D_{\gamma_z}(H^1(\Omega), z) = \{ z \} \),
Lemma 4.1.1. Assume that assumption \((A.1)\) is fulfilled, \(z_0 \in Z\) and \(f'(z_0) \not\in \sigma(-\Delta; \Omega)\). Then \(\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega)), 0) = (-1)^{\nu(f'(z_0))}\).

**Proof.** It is easy to see that \(\nabla^2 \Phi(z_0) = \text{Id} - (1 + f'(z_0))K\). Since \(f'(z_0) \not\in \sigma(-\Delta; \Omega)\), \(\nabla^2 \Phi(z_0)\) is an isomorphism. From the properties of the Leray-Schauder degree we get \(\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega)), 0) = \text{deg}_{\text{LS}}(\nabla^2 \Phi(z_0), B_{\gamma_0}(H^1(\Omega)), 0)\). The rest of the prove is a direct consequence of Lemma 3.1. □

Since the proof of the next lemma is similar to the proof of Lemma 4.1.1, we will omit it.

**Lemma 4.1.2.** Assume that assumptions \((A.1), (A.2)\) are satisfied and that \(f'(\infty) \not\in \sigma(-\Delta; \Omega)\). Then \(\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_\infty}(H^1(\Omega)), 0) = (-1)^{\nu(f'(\infty))}\).

Put the following assumptions:

\((A.3)\) \#\(Z\) \(<\infty\),

\((A.4)\) \(f'(z) \not\in \sigma(-\Delta; \Omega)\) for every \(z \in Z \cup \{\infty\}\).

Define \(Z_+ := \{z \in Z : f'(z) > 0\}\), \(Z_- := \{z \in Z : f'(z) < 0\}\).

Notice that if assumption \((A.4)\) is fulfilled, then \(Z_+ \cup Z_- = Z\).

In the next theorem we ensure the existence of nonconstant solutions of equation \((4.1.1)\).

**Theorem 4.1.1.** Suppose that assumptions \((A.1)-(A.4)\) are fulfilled. Moreover, assume that

1. if \(f'(\infty) < 0\), then there exists \(z_0 \in Z_+\) such that \(\nu(f'(z_0))\) is even,
2. if \(f'(\infty) > 0\) and \(\nu(f'(\infty))\) is odd, then there exists \(z_0 \in Z_+\) such that \(\nu(f'(z_0))\) is even,
3. if \(f'(\infty) > 0\) and \(\nu(f'(\infty))\) is even, then \(\#\{z \in Z_+ : \nu(f'(z))\text{ is even}\} \neq 1\).

Then there exists at least one nonconstant solution of equation \((4.1.1)\).

**Proof.** By the properties of the Leray-Schauder degree we obtain that

\[
\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_\infty}(H^1(\Omega)) \setminus \bigcup_{z \in Z} D_{\gamma_z}(H^1(\Omega), z), 0) = \\
\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_\infty}(H^1(\Omega)), 0) - \sum_{z \in Z} \text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z), 0).
\]

What is left is to show that

\[
\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_\infty}(H^1(\Omega)), 0) \neq \sum_{z \in Z} \text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z), 0).
\]

Suppose, contrary to our claim, that

\[
\text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_\infty}(H^1(\Omega)), 0) = \sum_{z \in Z} \text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z), 0). \tag{4.1.2}
\]

By Lemma 4.1.1 we obtain

\[
\sum_{z \in Z_-} \text{deg}_{\text{LS}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z), 0) = \#Z_-. \tag{4.1.3}
\]
Now put $Z_+^o := \{ z \in Z_+ : \nu(f'(z)) \text{ is odd} \}$ and $Z_+^e := \{ z \in Z_+ : \nu(f'(z)) \text{ is even} \}$. Then $Z_+ = Z_+^o \cup Z_+^e$ and $Z_+^o \cap Z_+^e = \emptyset$. Again from Lemma 4.1.1 it follows that

$$\sum_{z \in Z_+} \deg_{\text{LS}}(\nabla \Phi, B_{\gamma}(H^1(\Omega), z), 0) = \#Z_+^o - \#Z_+^e. \tag{4.1.4}$$

By Lemma 4.1.2 we have if $f'(\infty) < 0$, then $\deg_{\text{LS}}(\nabla \Phi, B_{\gamma}(H^1(\Omega), 0)) = 1$. Moreover, if $f'(\infty) > 0$ and $\nu(f'(\infty))$ is odd, then $\deg_{\text{LS}}(\nabla \Phi, B_{\gamma}(H^1(\Omega), 0)) = -1$.

Let assumption (1) or (2) be fulfilled. Then $\deg_{\text{LS}}(\nabla \Phi, B_{\gamma}(H^1(\Omega), 0)) = -\text{sign} f'(\infty)$. From this and equations (4.1.2)-(4.1.4) we obtain $\#Z_+^o - \#Z_+^e + \#Z_- = -\text{sign} f'(\infty)$. Moreover, it is easy to see that $\#Z_+ - \#Z_- = \text{sign} f'(\infty)$. Hence

$$\begin{cases} \#Z_+^o - \#Z_+^e + \#Z_- = -\text{sign} f'(\infty), \\
\#Z_+ - \#Z_- = \text{sign} f'(\infty). \end{cases}$$

We thus get $\#Z_+^o = 0$, a contradiction.

(3) By Lemma 4.1.2 we obtain $\deg_{\text{LS}}(\nabla \Phi, B_{\gamma}(H^1(\Omega), 0)) = 1$. Therefore

$$\begin{cases} \#Z_+^e - \#Z_+^o + \#Z_- = 1, \\
\#Z_+ - \#Z_- = 1, \end{cases}$$

which implies $\#Z_+^e = 1$, a contradiction. \hfill \Box

From now on we assume that

(A.5) $V = \mathbb{R}^n$ is a nontrivial orthogonal $SO(2)$-representation and that $\Omega \subset V$ is $SO(2)$-invariant.

**Remark 4.1.2.** Since $\Omega \subset V$ is $SO(2)$-invariant, $H^1(\Omega)$ is an orthogonal $SO(2)$-representation, with $SO(2)$-action defined by $(gu)(x) = u(gx)$, and $\Phi \in C^2_{SO(2)}(H^1(\Omega), \mathbb{R})$. Hence $\nabla \Phi \in C_{SO(2)}^1(H^1(\Omega), H^1(\Omega))$.

The following two lemmas are similar to 4.1.1, 4.1.2, respectively. Since $\nabla \Phi$ is $SO(2)$-invariant, instead of the Leray-Schauder degree we will apply the degree for $SO(2)$-equivariant gradient maps.

**Lemma 4.1.3.** Assume that assumptions (A.1), (A.5) are fulfilled. Fix $z_0 \in Z$ such that $f'(z_0) \not\in \sigma(-\Delta; \Omega)$. If $z_0 \in Z_+$, then

$$\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma z_0}(H^1(\Omega), z_0)) = \prod_{\lambda_i < f'(z_0)} \nabla_{SO(2)} - \deg(-\text{Id}, B_{\gamma z_0}(\nabla - \Delta(\lambda_i))) \in U(SO(2)).$$

Moreover, if $z_0 \in Z_-$, then $\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma z_0}(H^1(\Omega), z_0)) = 1 \in U(SO(2))$.

**Proof.** Since $z_0 \in H^1(\Omega)$ is a constant function, $B_{\gamma z_0}(H^1(\Omega), z_0) \subset H^1(\Omega)$ is $SO(2)$-invariant. Moreover, $\nabla \Phi$ is an $SO(2)$-equivariant operator of the form compact perturbation of the identity. Hence $\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma}(H^1(\Omega), z_0)) \in U(SO(2))$ is well-defined. It is clear that $\nabla^2 \Phi(z_0) = \text{Id} - (f'(z_0) + 1)K$ and that $\nabla^2 \Phi(z_0)$ is an isomorphism. From Theorem 2.1 we have

$$\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma z_0}(H^1(\Omega), z_0)) = \nabla_{SO(2)} - \deg(\nabla^2 \Phi(z_0), B_{\gamma z_0}(H^1(\Omega))).$$
The rest of the proof is a direct consequence of Lemma 3.2.

**Lemma 4.1.4.** Assume that assumptions (A.1), (A.2), (A.5) are satisfied and that $f'(\infty) \not\in \sigma(-\Delta; \Omega)$. Then

1. if $f'(\infty) > 0$, then
   \[
   \nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) = \prod_{\lambda_i < f'(\infty)} \nabla_{SO(2)} - \deg(-\text{Id}, B_{\gamma_{\infty}}(V_{-\Delta}(\lambda_i))) \in U(SO(2)),
   \]

2. if $f'(\infty) < 0$, then $\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) = \mathbb{1} \in U(SO(2))$.

**Proof.** Since $\nabla \Phi$ is an $SO(2)$-equivariant operator of the form compact perturbation of the identity and $\nabla^2 \Phi(\infty) = \text{Id} - (1 + f'(\infty))K$ is an isomorphism,

\[
\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) = \nabla_{SO(2)} - \deg(\nabla^2 \Phi(\infty), B_{\gamma_{\infty}}(H^1(\Omega))).
\]

The rest of the proof is a direct consequence of Lemma 3.2. □

The following corollary is an immediate consequence of Lemmas 2.1, 4.1.3, 4.1.4.

**Corollary 4.1.1.** If $z \in Z$ and assumptions of Lemma 4.1.3 are satisfied, then

1. if $H \in \Upsilon(SO(2))$ and $\nabla_{SO(2)} - \deg_H(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z)) \neq 0$, then
   \[
   \sign(\nabla_{SO(2)} - \deg_H(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z))) = (-1)^{\nu(f'(z))},
   \]

2. $\nabla_{SO(2)} - \deg_{SO(2)}(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z)) = (-1)^{\nu(f'(z))}$.

If $z = \infty$ and assumptions of Lemma 4.1.4 are fulfilled, then

1. if $H \in \Upsilon(SO(2))$ and $\nabla_{SO(2)} - \deg_H(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) \neq 0$, then
   \[
   \sign(\nabla_{SO(2)} - \deg_H(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega)))) = (-1)^{\nu(f'(\infty))},
   \]

2. $\nabla_{SO(2)} - \deg_{SO(2)}(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) = (-1)^{\nu(f'(\infty))}$.

Define $\lambda_0 = \min \{\lambda_i \in \sigma(-\Delta; \Omega) : \forall_{-\Delta}(\lambda_i) \text{ is a nontrivial } SO(2)\text{-representation}\}$. Moreover, for $z \in Z \cup \{\infty\}$ define $\forall(f'(z)) = \bigoplus_{\lambda_i < f'(z)} \forall_{-\Delta}(\lambda_i)$.

In the next three theorems we prove the existence of nonconstant solutions of equation (4.1.1). Since $\Omega \subset \mathcal{V}$ is $SO(2)$-invariant, $\nabla \Phi$ is $SO(2)$-equivariant. Therefore we use in the proofs the degree for $SO(2)$-equivariant gradient maps. It is worth to point out that we obtain the existence of nonconstant solutions of equation (4.1.1) also if the assumptions of Theorem 4.1.1 are not fulfilled.

**Theorem 4.1.2.** Suppose that assumptions (A.1)-(A.5) are fulfilled. Moreover, assume that $f'(\infty) < 0$ and that there exists $z_0 \in Z_+$ such that $\lambda_0 < f'(z_0)$. Then there exists at least one nonconstant solution of equation (4.1.1).
Proof. In view of Theorem 4.1.1, to complete the proof, it is enough to assume that $\nu(f'(z))$ is odd for all $z \in Z_+$. By the properties of the degree for $SO(2)$-equivariant gradient maps we obtain

$$\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) \cup \bigcup_{z \in Z} D_{\gamma_z}(H^1(\Omega), z) =$$

$$= \nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) - \sum_{z \in Z} \nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)).$$

Therefore, to complete the proof, it remains to prove that

$$\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) \neq \sum_{z \in Z} \nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)).$$

Suppose, contrary to our claim, that

$$\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) = \sum_{z \in Z} \nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)). \quad (4.1.5)$$

Since $\mathbb{V}_-\Delta(\lambda_0)$ is a nontrivial $SO(2)$-representation, there is $k' \in \mathbb{N}$ such that $\mathbb{V}_-\Delta(\lambda_0) = \mathbb{R}[1, k'] \oplus \mathbb{R}[1, k']^\perp$. From (4.1.5) we get

$$\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) = \sum_{z \in Z} \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)). \quad (4.1.6)$$

Since $f'(\infty) < 0$ and Lemma 4.1.4, we obtain

$$\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) = 0. \quad (4.1.7)$$

If $z \in Z_-$, then, by Lemma 4.1.3, we have

$$\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) = 0. \quad (4.1.8)$$

Taking into account (4.1.6), (4.1.7) and (4.1.8) we obtain

$$\sum_{z \in Z_+} \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) = 0. \quad (4.1.9)$$

Fix $z \in Z_+$. From Lemma 4.1.3 we have

$$\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) = \prod_{\lambda_i < f'(z)} \nabla_{SO(2)} - \deg(-Id, B_{\gamma_z}(\mathbb{V}_-\Delta(\lambda_i))) =$$

$$= \nabla_{SO(2)} - \deg(-Id, B_{\gamma_z}(\mathbb{V}(f'(z))).$$

By assumption $\nu(f'(z))$ is odd. Hence from Corollary 4.1.1 we obtain

$$\nabla_{SO(2)} - \deg_{SO(2)}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) = -1$$

and $\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) \leq 0$. Using the above and (4.1.9) we get $\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) = 0$ for all $z \in Z_+$. By the assumption there exists $z_0 \in Z_+$ such that $f'(z_0) > \lambda_0$. Therefore $\mathbb{V}(f'(z_0)) = \mathbb{R}[1, k'] \oplus \mathbb{R}[1, k']^\perp$. Finally, by Lemmas 2.1, 4.1.3, we obtain

$$\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_{z_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} - \deg_{Z_{k'}}(-Id, B_{\gamma_{z_0}}(\mathbb{V}(f'(z_0)))) \neq 0,$$

a contradiction. □
Theorem 4.1.3. Suppose that assumptions (A.1)-(A.5) are fulfilled, \( f'(\infty) > 0 \) and \( \nu(f'(\infty)) \) is odd. Additionally, assume that one of the following conditions is satisfied

1. there are \( z_0, z_1 \in \mathbb{Z}_+ \) such that \( f'(z_0) \geq f'(z_1) > \lambda_0 \) and \( f'(z_0) > f'(\infty) \),
2. there exists exactly one \( z_0 \in \mathbb{Z}_+ \) such that
   \( f'(z_0) > \lambda_0 \),
   \( \lambda_{i_0} \in \sigma(-\Delta; \Omega) \) such that \( f'(z_0) < \lambda_{i_0} < f'(\infty) \) (or \( f'(\infty) < \lambda_{i_0} < f'(z_0) \) and that \( V_{-\Delta}(\lambda_{i_0}) \) is a nontrivial \( \text{SO}(2) \)-representation,
3. there exists \( \lambda_{i_0} \in \sigma(-\Delta; \Omega) \) such that
   \( f'(z) < \lambda_{i_0} < f'(\infty) \) for all \( z \in \mathbb{Z}_+ \),
   \( \text{there exists } k' \in \mathbb{N} \) such that
      \( \forall_{-\Delta}(\lambda_{i_0}) = \mathbb{R}[1, k'] + \mathbb{R}[1, k']^{-} \),
      \( \mathbb{R}[1, k'] \nsubseteq \forall_{-\Delta}(\lambda_{i}) \) for \( \lambda_{i} \in \sigma(-\Delta; \Omega) \cap (-\infty, \lambda_{i_0}) \).

Then there exists at least one nonconstant solution of equation (4.1.1).

Proof. The proof is similar to that of Theorem 4.1.2. By the properties of the degree for \( \text{SO}(2) \)-equivariant gradient maps we obtain

\[
\nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_0}(H^1(\Omega)) \setminus \bigcup_{z \in \mathbb{Z}} D_{\gamma_k}(H^1(\Omega), z)) =
\]

\[
= \nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) - \sum_{z \in \mathbb{Z}} \nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_k}(H^1(\Omega), z)).
\]

It remains to prove that

\[
\nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) \neq \sum_{z \in \mathbb{Z}} \nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_k}(H^1(\Omega), z)).
\]

Suppose, contrary to our claim, that

\[
\nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) = \sum_{z \in \mathbb{Z}} \nabla_{\text{SO}(2)} - \deg(\nabla \Phi, B_{\gamma_k}(H^1(\Omega), z)).
\]

If \( z \in \mathbb{Z}_- \) and \( k \in \mathbb{N} \), then, by Lemma 4.1.3, we get \( \nabla_{\text{SO}(2)} - \deg_{Z_k} (\nabla \Phi, B_{\gamma_k}(H^1(\Omega), z)) = 0 \) and

\[
\nabla_{\text{SO}(2)} - \deg_{Z_k} (\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) = \sum_{z \in \mathbb{Z}_+} \nabla_{\text{SO}(2)} - \deg_{Z_k} (\nabla \Phi, B_{\gamma_k}(H^1(\Omega), z)). \quad (4.1.10)
\]

From Theorem 4.1.1 it follows that, to complete the proof, it suffices to consider the case \( \nu(f'(z)) \) is odd for all \( z \in \mathbb{Z}_+ \cup \{ \infty \} \). Therefore, by Corollary 4.1.1, we obtain that \( \nabla_{\text{SO}(2)} - \deg_{Z_k} (\nabla \Phi, B_{\gamma_k}(H^1(\Omega), z)) \leq 0 \) for all \( z \in \mathbb{Z}_+ \cup \{ \infty \} \) and \( k \in \mathbb{N} \).

1. Since \( V_{-\Delta}(\lambda_0) \) is a nontrivial \( \text{SO}(2) \)-representation there is \( k' \in \mathbb{N} \) such that \( \forall_{-\Delta}(\lambda_0) = \mathbb{R}[1, k'] + \mathbb{R}[1, k']^{-} \). Hence, by Lemma 2.1, we have

\[
\nabla_{\text{SO}(2)} - \deg_{Z_{k'}} (\nabla \Phi, B_{\gamma_{q_0}}(H^1(\Omega), z_0)), \nabla_{\text{SO}(2)} - \deg_{Z_{k'}} (\nabla \Phi, B_{\gamma_{q_1}}(H^1(\Omega), z_1)) < 0.
\]

Since \( f'(z_0) > f'(\infty) \), it follows that \( \forall(f'(\infty)) \subset \forall(f'(z_0)) \) and consequently, by Lemmas 2.1, 4.1.3, 4.1.4, we obtain

\[
\nabla_{\text{SO}(2)} - \deg_{Z_{k'}} (\nabla \Phi, B_{\gamma_{q_0}}(H^1(\Omega), z_0)) = \nabla_{\text{SO}(2)} - \deg_{Z_{k'}} (\nabla^2 \Phi(z_0), B_{\gamma_{q_0}}(H^1(\Omega), z_0)) =
\]
Taking together the above inequalities and (4.1.10) we obtain

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) \leq \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) = \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))). \]

Thus, by the above and (4.1.10), we obtain

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) \geq \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z_0)) > \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z_1)) + \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z_0) \geq \sum_{z \in Z_+} \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z)) = \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega)), \text{a contradiction.}) \]

(2) Since \( \mathcal{V}_{-\Delta}(\lambda_{i_0}) \) is a nontrivial \( SO(2) \)-representation, there is \( k' \in \mathbb{N} \) such that \( \mathcal{V}_{-\Delta}(\lambda_{i_0}) = \mathbb{R}[1, k'] \oplus \mathbb{R}[1, k'] \). Fix \( z \in Z \setminus \{z_0\} \). Since \( f'(z) < \lambda_0, \mathcal{V}(f'(z)) \) is a trivial \( SO(2) \)-representation, applying Lemmas 2.1, 4.1.3, we obtain

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z)) = 0. \]

Thus

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) = \sum_{z \in Z} \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z)) = \sum_{z \in Z_+} \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z)). \]

Let \( j_0, j_{\infty} \in \mathbb{N} \) be the largest integers such that

\[ \mathcal{V}(f'(z_0)) = \mathbb{R}[j_0, k'] \oplus \mathbb{R}[j_0, k'] \text{, and } \mathcal{V}(f'(\infty)) = \mathbb{R}[j_{\infty}, k'] \oplus \mathbb{R}[j_{\infty}, k'] \text{.} \]

Since \( f'(z_0) < \lambda_{i_0} < f'(\infty) \), we obtain \( j_0 < j_{\infty} \). Finally, by Lemmas 2.1, 4.1.3, we obtain

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) \neq \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z_0)), \]

a contradiction. The same proof remains valid if \( f'(\infty) < \lambda_{i_0} < f'(z_0) \).

(3) Since \( \mathbb{R}[1, k'] \not\subset \mathcal{V}(f'(z)) \) for every \( z \in Z_+ \) and Lemmas 2.1, 4.1.3,

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z)) = \nabla_{SO(2)} \deg_{Z_{\nu}}(-Id, B_{\gamma_{\nu}}(\mathcal{V}(f'(z))) = 0, \]

for every \( z \in Z \).

Thus, by the above and (4.1.10), we obtain

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) = \sum_{z \in Z_+} \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega), z)) = 0. \]

Since \( \mathcal{V}_{-\Delta}(\lambda_{i_0}) \subset \mathcal{V}(f'(\infty)), \mathbb{R}[1, k'] \subset \mathcal{V}_{\infty} \) and consequently

\[ \nabla_{SO(2)} \deg_{Z_{\nu}}(\Phi, B_{\gamma_{\nu}}(H^1(\Omega))) = \nabla_{SO(2)} \deg_{Z_{\nu}}(-Id, B_{\gamma_{\nu}}(\mathcal{V}(f'(\infty))) \neq 0, \]

a contradiction. \( \square \)

**Theorem 4.1.4.** Suppose that assumptions (A.1)-(A.5) are fulfilled, \( f'(\infty) > 0 \) and \( \nu(f'(\infty)) \) is even. Additionally assume that there exists \( z_0 \in Z_+ \) such that \( \nu(f'(z_0)) \) is even and one of the following conditions is fulfilled.
(1) there exist \( z_1, z_2 \in (Z_+ \cup \{\infty\}) \setminus \{z_0\} \) such that \( f'(z_1) \geq f'(z_2) > \lambda_0 \) and \( f'(z_1) > f'(z_0) \),

(2) there exists exactly one \( z_1 \in (Z_+ \cup \{\infty\}) \setminus \{z_0\} \) such that
   
   (a) \( f'(z_1) > \lambda_0 \),
   
   (b) there exists \( \lambda_{i_0} \in \sigma(-\Delta; \Omega) \) such that \( f'(z_1) < \lambda_{i_0} < f'(z_0) \) (or \( f'(z_0) < \lambda_{i_0} < f'(z_1) \)) and that \( \nabla_{-\Delta}(\lambda_{i_0}) \) is a nontrivial \( \text{SO}(2) \)-representation,

(3) there exists \( \lambda_{i_0} \in \sigma(-\Delta; \Omega) \) such that
   
   (a) \( f'(z) < \lambda_{i_0} < f'(z_0) \) for all \( z \in (Z_+ \cup \{\infty\}) \setminus \{z_0\} \),
   
   (b) there exists \( k' \in \mathbb{N} \) such that
      
      (i) \( \nabla_{-\Delta}(\lambda_{i_0}) = \mathbb{R}[1, k'] \oplus \mathbb{R}[1, k']^\perp \)
      
      (ii) \( \mathbb{R}[1, k'] \not\subset \nabla_{-\Delta}(\lambda_i) \) for \( \lambda_i \in \sigma(-\Delta; \Omega) \cap (-\infty, \lambda_{i_0}) \).

Then there exists at least one nonconstant solution of equation (4.1.1).

**Proof.** The proof is similar to that of Theorem 4.1.2. By the properties of the degree for \( \text{SO}(2) \)-equivariant gradient maps we obtain

\[
\nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega)) \setminus \bigcup_{z \in Z} D_{\gamma_z}(H^1(\Omega), z)) = \\
= \nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) - \sum_{z \in \mathbb{Z}} \nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)).
\]

It remains to prove that

\[
\nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) \neq \sum_{z \in \mathbb{Z}} \nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)).
\]

Suppose, contrary to our claim, that

\[
\nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) = \sum_{z \in \mathbb{Z}} \nabla_{\text{SO}(2)}\text{deg}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)).
\]

If \( z \in Z_- \) and \( k \in \mathbb{N} \), then, by Lemma 4.1.3, we get \( \nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) = 0 \) and \( \nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) = \sum_{z \in Z_+} \nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) \), which is equivalent to

\[
-\nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega), z_0)) = \\
= \sum_{z \in Z_+ \setminus \{z_0\}} \nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) - \nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))).
\]

(4.1.11)

Since \( \nu(f'(\infty)), \nu(f'(z_0)) \) are even and Corollary 4.1.1, we have

\[
-\nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega))) \leq 0, -\nabla_{\text{SO}(2)}\text{deg}_{Z_k}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega), z_0)) \leq 0.
\]

Notice that, in view of Theorem 4.1.1, to complete the proof it is enough to consider the case

\[
\{z \in Z_+ : \nu(f'(z)) \text{ is even}\} = \{z_0\}.
\]

(4.1.12)
(1) Let $z_1, z_2 \neq \infty$. Since $\mathbb{V}_-\Delta(\lambda_0)$ is a nontrivial $SO(2)$-representation, there is $k' \in \mathbb{N}$ such that $\mathbb{V}_-\Delta(\lambda_0) = \mathbb{R}[1, k'] \oplus \mathbb{R}[1, k']^\perp$. Taking into account that $f'(z_1), f'(z_2) > \lambda_0, \nu(f'(z_1)), \nu(f'(z_2))$ are odd and Corollary 4.1.1 we obtain

$$\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_1)), \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_2}(H^1(\Omega), z_2)) < 0.$$ 

Since $f'(z_1) > f'(z_0)$, it follows that $\mathbb{V}(f'(z_0)) \subset \mathbb{V}(f'(z_1))$ and consequently by Lemmas 2.1, 4.1.3, 4.1.4 we obtain

$$\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_1)) = \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla^2 \Phi(z_1), B_{\gamma_1}(H^1(\Omega), z_1)) =$$

$$= \nabla_{SO(2)} - \deg_{Z_{k'}}(-Id, B_{\gamma_1}(\mathbb{V}(f'(z_1)))) \leq -\nabla_{SO(2)} - \deg_{Z_{k'}}(-Id, B_{\gamma_1}(\mathbb{V}(f'(z_0)))) =$$

$$= -\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla^2 \Phi(z_0), B_{\gamma_1}(H^1(\Omega), z_0)) = -\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_0)).$$

Taking into account (4.1.11), (4.1.12), Corollary 4.1.1 and the above inequalities we obtain

$$-\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_0)) \geq \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_1)) >$$

$$> \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_2}(H^1(\Omega), z_2)) + \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_1)) \geq$$

$$\geq \sum_{z \in \mathbb{Z} \setminus \{z_0\}} \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_2}(H^1(\Omega), z)) - \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z)) =$$

$$= -\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_2}(H^1(\Omega), z_0)),$$

a contradiction. The same proof works for $z_1 = \infty$ or $z_2 = \infty$. The details are left to the reader.

(2) Assume that $z_1 \neq \infty$. Since $\mathbb{V}_-\Delta(\lambda_0)$ is a nontrivial $SO(2)$-representation, there is $k' \in \mathbb{N}$ such that $\mathbb{V}_-\Delta(\lambda_0) = \mathbb{R}[1, k'] \oplus \mathbb{R}[1, k']^\perp$. Fix $z \in (Z \cup \{\infty\}) \setminus \{z_0, z_1\}$. Since $f'(z) < \lambda_0$, $\mathbb{V}(f'(z))$ is a trivial $SO(2)$-representation and by Lemmas 2.1, 4.1.3 we have $\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z)) = 0$. Thus

$$-\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega), z_0)) = \sum_{z \in \mathbb{Z} \setminus \{z_0\}} \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_2}(H^1(\Omega), z)) -$$

$$-\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_2}(H^1(\Omega), z_0)) = \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_1)).$$

Let $j_0, j_1 \in \mathbb{N}$ be the largest integers such that

$$\mathbb{V}(f'(z_0)) = \mathbb{R}[j_0, k'] \oplus \mathbb{R}[j_0, k']^\perp,$$

and $\mathbb{V}(f'(z_1)) = \mathbb{R}[j_1, k'] \oplus \mathbb{R}[j_1, k']^\perp$. Since $f'(z_1) < \lambda_0 < f'(z_0)$, we obtain $j_1 < j_0$. Finally, by Lemmas 2.1, 4.1.3, we obtain

$$-\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega), z_0)) \neq \nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z_1)),$$

a contradiction. The same proof remains valid if $z_1 = \infty$ or $f'(z_0) < \lambda_0 < f'(z_1)$. The details are left to the reader.

(3) Since $\mathbb{R}[1, k'] \not\subset \mathbb{V}(f'(z))$ for every $z \in (Z_+ \cup \{\infty\}) \setminus \{z_0\}$ and Lemmas 2.1, 4.1.3, $\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_1}(H^1(\Omega), z)) = \nabla_{SO(2)} - \deg_{Z_{k'}}(-Id, B_{\gamma_1}(\mathbb{V}(f'(z)))) = 0$, for every $z \in (Z_+ \cup \{\infty\}) \setminus \{z_0\}$. Thus, by the above and (4.1.11), we obtain

$$-\nabla_{SO(2)} - \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_0}(H^1(\Omega), z_0)) =$$
\[= \sum_{z \in Z_+ \setminus \{z_0\}} \nabla_{SO(2)} - \deg_{\mathbb{Z}_+} (\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z)) - \nabla_{SO(2)} - \deg_{\mathbb{Z}_+} (\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) = 0.\]

Since \(\nabla_{-\Delta} (\lambda_0) \subset \mathbb{V}(f'(z_0)), \mathbb{R}[1, k'] \subset \mathbb{V}(f'(z_0))\) and consequently

\[\nabla_{SO(2)} - \deg_{\mathbb{Z}_+} (\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z_0)) = \nabla_{SO(2)} - \deg_{\mathbb{Z}_+} (\nabla \Phi, B_{\gamma_z}(H^1(\Omega), z_0)) \neq 0,\]
a contradiction. \(\square\)

**Remark 4.1.3.** Notice that in Theorems 4.1.2-4.1.4 the degree for \(SO(2)\)-equivariant gradient maps cannot be replaced with the Leray-Schauder degree, since it vanishes. In fact, under assumptions of these theorems it can happen that

\[\deg_{\mathbb{L}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) - \sum_{z \in Z} \deg_{\mathbb{L}}(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) = 0 \in \mathbb{Z}.\]

and that

\[\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) - \sum_{z \in Z} \nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_z}(H^1(\Omega))) \neq \Theta \in U(SO(2)).\]

In other words we obtain the existence of nonconstant solution of equation (4.1.1) in the situation when the Leray-Schauder degree is not applicable i.e. the assumptions of Theorem 4.1.1 are not fulfilled.

In the rest of this section we consider a degenerate case i.e. we allow \(f'(z_0) \in \sigma(-\Delta; \Omega)\) for some \(z_0 \in Z \cup \{\infty\}\). To compute a local index of a degenerate isolated critical point of \(\Phi\) we combine the splitting lemmas (Lemmas 3.2, 3.3 of [5]) and the product formula for the degree for \(SO(2)\)-equivariant gradient maps, Theorem 2.2.

The following lemma is a consequence of splitting lemmas of [5].

**Lemma 4.1.5.** Assume that assumptions (A.1), (A.5) are satisfied. Fix \(z_0 \in Z \cup \{\infty\}\) such that \(f'(z_0) \in \sigma(-\Delta; \Omega)\) and \(z_0 \in H^1(\Omega)\) is an isolated critical point of \(\Phi\). Then there exist \(\alpha_0 > 0\) and \(\varphi \in C^2_{SO(2)}(\nabla_{-\Delta}(f'(z_0)), \mathbb{R})\) such that \(0 \in \nabla_{-\Delta}(f'(z_0))\) is an isolated critical point of \(\varphi\) and that

\[\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\gamma_{z_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} - \deg(\nabla \varphi, B_{\alpha_0}(\nabla_{-\Delta}(f'(z_0)))) * \prod_{\lambda_i < f'(z_0)} \nabla_{SO(2)} - \deg(-\text{Id}, B_{\alpha_0}(\nabla_{-\Delta}(\lambda_i))).\]

**Proof.** Fix \(z_0 \in Z\) and remind that \(\mathcal{K} : H^1(\Omega) \to H^1(\Omega)\) is an \(SO(2)\)-equivariant, self-adjoint, compact operator such that \(\langle \mathcal{K} u, v \rangle_{H^1(\Omega)} = \int_{\Omega} u(x)v(x)dx\). Set \(L = (1 + f'(z_0))\mathcal{K}\), \(V_0 = \ker \text{Id} - L\) and \(W_0 = V_0^+ = \text{im} \text{Id} - L\). It is easy to see that \(\nabla^2 \Phi(z_0) = \text{Id} - L\) and \(V_0 = \nabla_{-\Delta}(f'(z_0))\). Define \(\nabla \eta_0 : H^1(\Omega) \to H^1(\Omega), \nabla \eta_0 = \nabla \Phi - (\text{Id} - L)\). Then \(\nabla \eta_0\) is a compact, \(SO(2)\)-equivariant operator and \(|\nabla \eta_0(u)| = o(|u|)\) as \(|u| \to 0\). Now applying Lemma 3.2 of [5] we obtain \(\alpha_0 > 0\) and \(\varphi \in C^2_{SO(2)}(\nabla_{-\Delta}(f'(z_0)), \mathbb{R})\) with isolated critical point at the origin and such that

\[\nabla_{SO(2)} - \deg(\nabla \Phi, B_{\alpha_0}(H^1(\Omega))) = \nabla_{SO(2)} - \deg((\nabla \varphi, (\text{Id} - L_0)|W_0\), B_{\alpha_0}(V_0)) \times B_{\alpha_0}(W_0)).\]
Finally, combining Theorem 2.2 with a slightly modified version of Lemma 3.2 (instead of the operator $\nabla_u \Psi(\cdot, \lambda)$ it is enough to consider the operator $\nabla^2 \Phi(z_0)|_{W_0}$), we obtain
\[
\nabla_{SO(2)} \cdot \deg((\nabla \varphi, (I - L_0)|_{W_0}), B_{\varphi_0}(V_0)) = \nabla_{SO(2)} \cdot \deg((\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)))) \ast \nabla_{SO(2)} \cdot \deg((I - H)|_{W_0}, B_{\gamma_{\lambda_{z_0}}}(W_0)) = \nabla_{SO(2)} \cdot \deg((\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)))) \ast \prod_{\lambda_i < z'(z_0)} \nabla_{SO(2)} \cdot \deg(-I \ast B_{\gamma_{\lambda_{z_0}}}(\nabla_{-\Delta}(\lambda_i)));
\]
which completes the proof.

The same proof remains valid for $z_0 = \infty$ but instead of Lemma 3.2 of [5] we must use Lemma 3.3 of [5]. The details are left to the reader. 

**Corollary 4.1.2.** Fix $z_0 \in Z \cup \{\infty\}$ satisfying assumptions of Lemma 4.1.5 and $k' \in \mathbb{N}$. Assume that

(1) $\mathbb{R}[1, k'] \not\subset \nabla(f'(z_0))$,

(2) $\nabla_{-\Delta}(f'(z_0)) \in \emptyset$.

Then $\nabla_{SO(2)} \cdot \deg_{Z_{k'}}(\nabla \varphi, B_{\gamma_{\varphi_0}}(H^1(\Omega), z_0)) = 0$.

**Proof.** Take $\alpha_0 > 0$ and $\nabla \varphi$ as in Lemma 4.1.5. Then
\[
\nabla_{SO(2)} \cdot \deg(\nabla \varphi, B_{\gamma_{\varphi_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} \cdot \deg((\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)))) \ast \nabla_{SO(2)} \cdot \deg(-I \ast B_{\gamma_{\varphi_0}}(\nabla(f'(z_0))))).
\]

By (1) and Lemma 2.1 we have $\nabla_{SO(2)} \cdot \deg_{Z_{k'}}(-I \ast B_{\gamma_{\varphi_0}}(\nabla(f'(z_0)))) = 0$. By (2) and Remark 2.2 we obtain $\nabla_{SO(2)} \cdot \deg_{Z_{k'}}((\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)))) = 0$. The rest of the proof is a direct consequence of product formula (2.2). 

**Corollary 4.1.3.** Fix $z_0 \in Z \cup \{\infty\}$ satisfying assumptions of Lemma 4.1.5. If moreover, $\nabla_{SO(2)}(f'(z_0))^\text{SO}(2) = \{0\}$, then
\[
\nabla_{SO(2)} \cdot \deg_{Z_{k}}(\nabla \varphi, B_{\gamma_{\varphi_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} \cdot \deg_{Z_{k}}(-I \ast B_{\gamma_{\varphi_0}}(\nabla(f'(z_0))))
\]

for all $k' \in \mathbb{N}$ such that $\nabla_{-\Delta}(f'(z_0)) \in \emptyset$.

**Proof.** Take $\alpha_0 > 0$ and $\nabla \varphi$ as in Lemma 4.1.5. Then
\[
\nabla_{SO(2)} \cdot \deg(\nabla \varphi, B_{\gamma_{\varphi_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} \cdot \deg((\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)))) \ast \nabla_{SO(2)} \cdot \deg(-I \ast B_{\gamma_{\varphi_0}}(\nabla(f'(z_0))))).
\]

Since $\nabla_{-\Delta}(f'(z_0))^\text{SO}(2) = \{0\}$, $\nabla_{SO(2)} \cdot \deg_{SO(2)}(\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)), 0)) = 1$. Moreover, since $\nabla_{-\Delta}(f'(z_0)) \in \emptyset$ and Remark 2.2, $\nabla_{SO(2)} \cdot \deg_{Z_{k}}(\nabla \varphi, B_{\gamma_{\varphi_0}}(\nabla_{-\Delta}(f'(z_0)), 0)) = 0$. The rest of the proof is a direct consequence of formula (2.2). 

We can now proceed the analog of Theorems 4.1.2, 4.1.3, 4.1.4. It is worth to point out that in this theorem we allow $\bigcup_{z \in Z \cup \{\infty\}} \{f'(z)\} \cap \sigma(-\Delta; \Omega) \neq \emptyset$.

**Theorem 4.1.5.** Let assumptions (A.1)-(A.3), (A.5) be fulfilled. Moreover, assume that there are $z_0 \in Z \cup \{\infty\}$, $\lambda_{i_0} \in \sigma(-\Delta; \Omega)$ and $k' \in \mathbb{N}$ such that

(1) either $f'(z_0) \not\in \sigma(-\Delta; \Omega)$ or $\nabla_{SO(2)}(f'(z_0))^\text{SO}(2) = \{0\}$ and $\nabla_{-\Delta}(f'(z_0))Z_{k'} = \emptyset$,
Then there exists at least one nonconstant solution of equation (4.1.1).

Proof. Without loss of generality we can assume that elements of $Z \cup \{\infty\}$ are isolated critical points of the potential $\Phi$. To complete the proof it is enough to show that

$$\nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) \neq \sum_{z \in Z} \nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z)).$$

Suppose, contrary to our claim, that

$$\nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega))) = \sum_{z \in Z} \nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z)). \quad (4.1.13)$$

Fix $z \in (Z \cup \{\infty\}) \setminus \{z_0\}$. By assumptions (2), (4) we obtain that $\nabla_{-\Delta}(\lambda_i)_{z_{k'}} = \emptyset$ for all $\lambda_i \in \sigma(-\Delta; \Omega) \cap (-\infty, f'(z))$. Therefore $\mathbb{R}[1, k'] \not\subset \nabla(f'(z))$ and if $f'(z) \in \sigma(-\Delta; \Omega)$, then $\nabla_{-\Delta}(f'(z))_{z_{k'}} = \emptyset$. Hence, from Corollary 4.1.2, we obtain

$$\nabla_{SO(2)} \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z)) = 0, \quad (4.1.14)$$

for all $z \in (Z \cup \{\infty\}) \setminus \{z_0\}$.

If $f'(z_0) \not\in \sigma(-\Delta; \Omega)$, then, by Lemmas 4.1.3 ($z_0 \in Z$) or Lemma 4.1.4 ($z_0 = \infty$), we get

$$\nabla_{SO(2)} \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_{z_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} \deg_{Z_{k'}}(-\text{Id}, B_{\gamma_{z_0}}(\nabla(f'(z_0)), z_0)).$$

If $f'(z_0) \in \sigma(-\Delta; \Omega)$, then, by assumption (1) and Corollary 4.1.3, we get

$$\nabla_{SO(2)} \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_{z_0}}(H^1(\Omega), z_0)) = \nabla_{SO(2)} \deg_{Z_{k'}}(-\text{Id}, B_{\gamma_{z_0}}(\nabla(f'(z_0)), z_0)).$$

Finally, since $\lambda_{i_0} < f'(z_0)$ and $\mathbb{R}[1, k'] \subset \nabla_{-\Delta}(\lambda_{i_0})$, we obtain $\mathbb{R}[1, k'] \subset \nabla(f'(z_0))$. Thus, by Lemma 2.1, we obtain $\nabla_{SO(2)} \deg_{Z_{k'}}(-\text{Id}, B_{\gamma_{z_0}}(\nabla(f'(z_0)), z_0)) \neq 0$ and consequently

$$\nabla_{SO(2)} \deg_{Z_{k'}}(\nabla \Phi, B_{\gamma_{z_0}}(H^1(\Omega), z_0)) \neq 0. \quad (4.1.15)$$

Combining (4.1.14) with (4.1.15) we get

$$\nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{\infty}}(H^1(\Omega), 0)) - \sum_{z \in Z} \nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{z}}(H^1(\Omega), z)) =$$

$$= -\nabla_{SO(2)} \deg(\nabla \Phi, B_{\gamma_{z_0}}(H^1(\Omega), z_0)) \neq 0,$$

contrary to (4.1.13).

□

Remark 4.1.4. Let us notice that in the above theorem the same proof works for assumption (4) replaced by assumptions

1. $\mathbb{R}[1, k'] \not\subset \nabla_{-\Delta}(\lambda_i)$ for all $\lambda_i \in \sigma(-\Delta; \Omega) \cap (-\infty, \lambda_{i_0})$,
2. $\nabla_{-\Delta}(f'(z))_{z_{k'}} = \emptyset$ for all $z \in (Z \cup \{\infty\}) \cap \sigma(-\Delta; \Omega)$.

The details are left to the reader.
4.2. Continuation of solutions. Consider a family of equations of the form

\[
\begin{aligned}
-\Delta u &= f(u, \lambda) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \), \( f(\cdot, \lambda) \) satisfies condition (A.1) for every \( \lambda \in \mathbb{R} \) and \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^1 \)-boundary.

In this section we study continuation of nonconstant solutions of family (4.2.1).

Remark 4.2.1. Consider a functional \( \Phi \in C^2(H^1(\Omega) \times \mathbb{R}, \mathbb{R}) \) defined as follows

\[
\Phi(u, \lambda) = \int_\Omega |\nabla u(x)|^2 - f(u(x), \lambda)dx.
\]

Define \( Z_0 = (f(\cdot, 0))^{-1}(0) \) and assume that

1. \( \#Z_0 < \infty \),
2. all the elements of \( Z_0 \cup \{\infty\} \) are isolated critical points of \( \Phi(\cdot, 0) \).

Define an open bounded set \( U \) in the following way

\[
U = B_{\gamma_\infty}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z).
\]

Since \((\nabla_u \Phi(\cdot, 0))^{-1}(0) \cap \partial U = \emptyset, (\nabla_u \Phi(\cdot, 0))^{-1}(0) \cap \partial U = \emptyset\). Therefore, by the properties of the Leray-Schauder degree, we obtain

\[
\deg_{LS}(\nabla_u \Phi(\cdot, 0), U, 0) = \deg_{LS}(\nabla_u \Phi(\cdot, 0), B_{\gamma_\infty}(H^1(\Omega)), 0) - \sum_{z \in Z_0} \deg_{LS}(\nabla_u \Phi(\cdot, 0), B_{\gamma_z}(H^1(\Omega), z), 0).
\]

If moreover, \( \mathbb{R}^n \) is a finite-dimensional \( SO(2) \)-representation and \( \Omega \) is \( SO(2) \)-invariant, then \( \Phi \in C^2_{SO(2)}(H^1(\Omega) \times \mathbb{R}, \mathbb{R}) \) and \( U \) is \( SO(2) \)-invariant. Therefore, by the properties of the degree for \( SO(2) \)-equivariant gradient maps, we obtain

\[
\nabla_{SO(2)} - \deg(\nabla \Phi_u(\cdot, 0), U) = \nabla_{SO(2)} - \deg(\nabla \Phi_u(\cdot, 0), B_{\gamma_\infty}(H^1(\Omega))) - \sum_{z \in Z_0} \nabla_{SO(2)} - \deg(\nabla \Phi_u(\cdot, 0), B_{\gamma_z}(H^1(\Omega), z)).
\]

Theorem 4.2.1. Fix \( f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and assume that \( f(\cdot, 0) \) satisfies assumptions of Theorem 4.1.1. Then there exist closed connected sets \( C^\pm \) such that

\[
C^- \subset (H^1(\Omega) \times (-\infty, 0]) \cap (\nabla_u \Phi)^{-1}(0) \quad \text{and} \quad C^+ \subset (H^1(\Omega) \times [0, +\infty)) \cap (\nabla_u \Phi)^{-1}(0).
\]

Moreover, for \( C = C^\pm \)

(i) \( C \cap ((B_{\gamma_\infty}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z)) \times \{0\}) \neq \emptyset \),

(ii) either \( C \) is unbounded or \( C \cap (Z_0 \times \{0\}) \neq \emptyset \).
Proof. Repeating the reasoning from the proof of Theorem 4.1.1 we obtain
\[ \deg_{LS}(\nabla \Phi(\cdot, 0), B_{\gamma_{\infty}}(H^1(\Omega)), 0) \neq \sum_{z \in Z_0} \deg_{LS}(\nabla \Phi(\cdot, 0), B_{\gamma_z}(H^1(\Omega), z), 0). \]
Define \( U = B_{\gamma_{\infty}}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z) \) and notice that \( \deg_{LS}(\nabla \Phi(\cdot, 0), U, 0) \neq 0. \)
Applying Theorem 2.5 we obtain the existence of closed connected sets \( C \) such that
\[ C^- \subset (H^1(\Omega) \times (-\infty, 0]) \cap (\nabla \Phi)^{-1}(0), \]
\[ C^+ \subset (H^1(\Omega) \times [0, +\infty)) \cap (\nabla \Phi)^{-1}(0), \]
\( C = C^\pm \) satisfies (i) and either \( C \) is unbounded or else \( C \cap ((H^1(\Omega) \setminus cl(U)) \times \{0\}) \neq \emptyset. \)
By definition \( (H^1(\Omega) \setminus cl(U)) \cap (\nabla \Phi)^{-1}(0) \subset \bigcup_{z \in Z_0} B_{\gamma_z}(H^1(\Omega), z) \). On the other hand
\[ \bigcup_{z \in Z_0} B_{\gamma_z}(H^1(\Omega), z) \cap (\nabla \Phi(\cdot, 0))^{-1}(0) = Z_0, \]
which completes the proof.

Theorem 4.2.2. Fix \( f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and assume that \( f(\cdot, 0) \) satisfies assumptions of one of Theorems 4.1.2, 4.1.2, 4.1.4. Then there exist closed connected sets \( C^\pm \) such that
\[ C^- \subset (H^1(\Omega) \times (-\infty, 0]) \cap (\nabla \Phi)^{-1}(0) \]
\[ C^+ \subset (H^1(\Omega) \times [0, +\infty)) \cap (\nabla \Phi)^{-1}(0). \]
Moreover, for \( C = C^\pm \)
(i) \( C \cap ((B_{\gamma_{\infty}}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z)) \times \{0\}) \neq \emptyset, \)
(ii) either \( C \) is unbounded or \( C \cap (Z_0 \times \{0\}) \neq \emptyset. \)
Proof. Repeating the reasoning from the proofs of Theorems 4.1.2-4.1.4 we obtain
\[ \nabla_{SO(2)} \deg(\nabla \Phi(\cdot, 0), B_{\gamma_{\infty}}(H^1(\Omega))) \neq \sum_{z \in Z_0} \nabla_{SO(2)} \deg(\nabla \Phi(\cdot, 0), B_{\gamma_z}(H^1(\Omega), z)). \]
Set \( U = B_{\gamma_{\infty}}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z) \). Notice that by Remark 4.2.1 we obtain
\[ \nabla_{SO(2)} \deg(\nabla \Phi(\cdot, 0), U) \neq \Theta. \]
The rest of the proof is a direct consequence of Theorem 2.5.

Theorem 4.2.3. Assume that \( f(\cdot, 0) \) satisfies assumptions of Theorem 4.1.5. Then there exists infinite sequence of nonconstant solutions of equation (4.2.1) with \( \lambda = 0 \) converging to some \( z \in Z_0 \) or there exist closed connected sets \( C^\pm \) such that
\[ C^- \subset (H^1(\Omega) \times (-\infty, 0]) \cap (\nabla \Phi)^{-1}(0) \]
\[ C^+ \subset (H^1(\Omega) \times [0, +\infty)) \cap (\nabla \Phi)^{-1}(0). \]
Moreover, for \( C = C^\pm \)
(i) \( C \cap ((B_{\gamma_{\infty}}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z)) \times \{0\}) \neq \emptyset, \)
(ii) either \( C \) is unbounded or \( C \cap (Z_0 \times \{0\}) \neq \emptyset. \)
Proof. Suppose that doesn’t exist a sequence of nonconstant solutions of equation (4.2.1) with \(\lambda = 0\) converging to some point in \(Z_0\). Then all the points \(z \in Z_0\) are isolated critical points of \(\Phi(\cdot, 0)\). Repeating the reasoning from the proof of Theorem 4.1.5 we obtain

\[
\nabla_{SO(2)} - \text{deg}(\nabla_u \Phi(\cdot, 0), B_{\gamma_\infty}(H^1(\Omega))) \neq \sum_{z \in Z_0} \nabla_{SO(2)} - \text{deg}(\nabla_u \Phi(\cdot, 0), B_{\gamma_z}(H^1(\Omega), z)).
\]

We set \(\mathcal{U} = B_{\gamma_\infty}(H^1(\Omega)) \setminus \bigcup_{z \in Z_0} D_{\gamma_z}(H^1(\Omega), z)\). Applying Remark 4.2.1 we obtain the following \(\nabla_{SO(2)} - \text{deg}(\nabla \Phi, \mathcal{U}) \neq \Theta\). The rest of the proof is a direct consequence of Theorem 2.5. \(\square\)

5. Examples

In this section we illustrate the abstract results proved in the previous section.

Define \(V_1 = \mathbb{R}[1, 1], V_2 = V_1 \oplus \mathbb{R}[1, 0]\) and denote by \(\Omega_1 \subset V_1\) an open disc of radius one in \(V_1\) and \(\Omega_2 = \Omega_1 \times (0, 1) \subset V_2\). Since \(SO(2)\)-representations \(V_1, V_2\) are orthogonal, sets \(\Omega_1, \Omega_2\) are \(SO(2)\)-invariant. First we remind some standard facts about \(\sigma(-\Delta, \Omega_i), i = 1, 2\).

Throughout this section we assume that \(k, n \in \mathbb{N} \cup \{0\}\). Moreover, if \(k \in \mathbb{N}\), then \(n \in \mathbb{N}\) and by \(x_{kn}\) we denote the \(n\)-th solution of \(J_k'(x) = 0\) in \((0, +\infty)\), where \(J_k\) is an \(k\)-th Bessel function. If \(k = 0\), then \(n \in \mathbb{N} \cup \{0\}\) and by \(x_{0n}\) we denote the \(n\)-th solution of \(J_0'(x) = 0\) in \([0, +\infty)\). Notice that \(x_{00} = 0\).

**Lemma 5.1.** ([11]) Under the above assumptions

1. \(\sigma(-\Delta, \Omega_1) = \{\lambda_{kn} = x_{kn}^2\}_{k=1,n=1}^\infty \cup \{\lambda_{0n} = x_{0n}^2\}_{n=0}^\infty\); with corresponding eigenvectors in spherical coordinates given by

   \[
   \begin{align*}
   (a) \quad & \text{if } k > 0, \text{ then } n > 0 \text{ and } \lambda_{kn} \longrightarrow v_{kn}(r, \phi) = J_k(x_{kn} r) \left\{ \begin{array}{l}
   \cos k \phi, \\
   \sin k \phi,
   \end{array} \right. \\
   (b) \quad & \text{if } k = 0, \text{ then } \lambda_{0n} \longrightarrow v_{0n}(r, \phi) = J_0(x_{0n} r).
   \end{align*}
   \]

2. \(\sigma(-\Delta, \Omega_2) = \{\lambda_{knj} = (\pi n)^2 + x_{kj}^2\}_{k=1,n=0,j=1}^\infty \cup \{\lambda_{0nj} = (\pi n)^2 + x_{0j}^2\}_{n=0,j=0}^\infty\); with corresponding eigenvectors in cylindrical coordinates given by

   \[
   \begin{align*}
   (a) \quad & \text{if } k > 0, \text{ then } j > 0 \text{ and } \lambda_{knj} \longrightarrow v_{knj}(r, \phi, z) = \cos(n \pi z) J_k(x_{kj} r) \left\{ \begin{array}{l}
   \cos k \phi, \\
   \sin k \phi,
   \end{array} \right. \\
   (b) \quad & \text{if } k = 0, \text{ then } \lambda_{0nj} \longrightarrow v_{0nj}(r, \phi, z) = \cos(n \pi z) J_0(x_{0j} r).
   \end{align*}
   \]

In the next lemma we show some properties of zeros of derivatives of Bessel functions.

**Lemma 5.2.** Under the above assumptions

1. \(0 = x_{00} < x_{01} < x_{02} < \ldots\),
2. \(0 < x_{k1} < x_{k2} < x_{k3} < \ldots\), for \(k \in \mathbb{N}\),
3. \(x_{11} < x_{21} < x_{31} < \ldots\).

Applying Lemmas 5.1, 5.2 we obtain the following corollary.

**Corollary 5.1.** Under the above assumptions

1. \(\lambda_{11} < \lambda_{21} < \ldots\),
2. \(\lambda_{101} < \lambda_{201} < \ldots\).
Lemma 5.3. If \( k \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \) and \( \lambda_{kn} \in \sigma(-\Delta; \Omega_1) \), then \( \mathbb{R}[1,k] \subset V_-(\lambda_{kn}) \). Additionally, \( V_-(\lambda_{00}) = \mathbb{R}[1,0] \).

**Proof.** First of all notice that from Lemma 5.1 we obtain

(1) if \( k > 0 \), then \( \text{span}_x \{ J_k(x_{kn}r) \cos k\phi, J_k(x_{kn}r) \sin k\phi \} \subset V_-(\lambda_{kn}) \),

(2) if \( k = 0 \) and \( n > 0 \), then \( \text{span}_x \{ J_0(x_{0n}r) \} \subset V_-(\lambda_{0n}) \),

(3) if \( k = 0 \) and \( n = 0 \), then \( \text{span}_x \{ v_{00} \} = V_-(\lambda_{00}) \).

Since the \( SO(2) \)-action \( SO(2) \times \mathbb{R}^1(\Omega_1) \rightarrow \mathbb{R}^1(\Omega_1) \) is given by

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
0 \\
r, \phi
\end{pmatrix}
= u(r, \phi) = u(r, \phi + \theta),
\]

it is easy to check that

(1) \( \text{span}_x \{ J_k(x_{kn}r) \cos k\phi, J_k(x_{kn}r) \sin k\phi \} \approx \mathbb{R}[1,k] \),

(2) \( \text{span}_x \{ J_0(x_{0n}r) \} \approx \mathbb{R}[1,0] \),

(3) \( \text{span}_x \{ v_{00} \} \approx \mathbb{R}[1,0] \),

which completes the proof. \( \square \)

The corollary below is a consequence of Lemma 5.3.

**Corollary 5.2.** If \( \lambda \in \sigma(-\Delta, \Omega_1) \) and

\[
\{(k, n) \in (\mathbb{N} \cup \{0\})^2 : \lambda_{kn} \in \sigma(-\Delta, \Omega_1) \text{ and } \lambda_{kn} = \lambda \} = \{(k_1, n_1), \ldots, (k_s, n_s)\},
\]

then \( V_-(\lambda) \approx \mathbb{R}[1,k_1] \oplus \mathbb{R}[1,k_2] \oplus \cdots \oplus \mathbb{R}[1,k_s] \).

Moreover, if \( a > 0 \) and \( \nu(a) = \sum_{\lambda_{kn} < a} \dim V_-(\lambda_{kn}) \), then

\( \nu(a) \) is even iff \( \# \{ \lambda_{0n} : \lambda_{0n} < a \} \) is even.

In Lemma 5.4 we describe eigenspaces of \( -\Delta \) corresponding to eigenvalues \( \sigma(-\Delta; \Omega_2) \) as \( SO(2) \)-representations.

**Lemma 5.4.** If \( k, n, j \in \mathbb{N} \cup \{0\} \) and \( \lambda_{knj} \in \sigma(-\Delta; \Omega_2) \), then \( \mathbb{R}[1,k] \subset V_-(\lambda_{knj}) \). Additionally, \( V_-(\lambda_{000}) = \mathbb{R}[1,0] \).

**Proof.** In fact the proof is the same as the proof of Lemma 5.3. The details are left to the reader. \( \square \)

The following corollary is a direct consequence of Lemmas 5.4.

**Corollary 5.3.** If \( \lambda \in \sigma(-\Delta, \Omega_2) \) and

\[
\{(k, n, j) \in (\mathbb{N} \cup \{0\})^3 : \lambda_{knj} \in \sigma(-\Delta, \Omega_2) \text{ and } \lambda_{knj} = \lambda \} = \{(k_1, n_1, j_1), \ldots, (k_s, n_s, j_s)\},
\]

then \( V_-(\lambda) \approx \mathbb{R}[1,k_1] \oplus \mathbb{R}[1,k_2] \oplus \cdots \oplus \mathbb{R}[1,k_s] \).

Moreover, if \( a > 0 \) and \( \nu(a) = \sum_{\lambda_{knj} < a} \dim V_-(\lambda_{knj}) \), then

\( \nu(a) \) is even iff \( \# \{ \lambda_{0nj} : \lambda_{0nj} < a \} \) is even.
Remark 5.1. For \( i = 1, 2 \) let \( \lambda_0(\Omega_i) \) be the smallest eigenvalue in \( \sigma(-\Delta, \Omega_i) \), such that \( V_{-\Delta}(\lambda_0(\Omega_i)) \) is a nontrivial \( SO(2) \)-representation. It clear that
\[
\lambda_0(\Omega_1) = \lambda_{11} = x_{11}^2 = \lambda_{01} = \lambda_0(\Omega_2).
\]

The proof of the lemma below is a direct consequence of estimations from [17, Section 15.3, p.486].

Lemma 5.5. For every \( k \in \mathbb{N} \), \( \lambda_{k_1} \in \sigma(-\Delta, \Omega_1) \) and \( \lambda_{k_01} \in \sigma(-\Delta, \Omega_2) \) we have
\[
k(k + 2) < \lambda_{k_1} < 2k(k + 1), \quad k(k + 2) < \lambda_{k_01} < 2k(k + 1).
\]
Consequently, \( 3 < \lambda_0(\Omega_1) = \lambda_0(\Omega_2) < 4 \).

Example 5.1. Consider equation
\[
\begin{align*}
-\Delta u &= f(u) \quad \text{in } \Omega_1, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega_1,
\end{align*}
\]
where \( f \) satisfies the following assumptions:
1. \( f \in C^1(\mathbb{R}, \mathbb{R}) \),
2. \( |f'(x)| \leq a + b|x|^q \) for some \( a, b > 0, q \in \mathbb{N} \),
3. \( f(t) = f'(\infty)t + o(|t|) \) where \( |t| \to \infty \),
4. \( 50 < f'(\infty) < 99 \),
5. \( \#Z < \infty \), where \( Z = f^{-1}(0) \),
6. \( f'(z) \not\in \sigma(-\Delta, \Omega_1) \), for all \( z \in Z \),
7. there are \( z_0, z_1 \in Z \) such that \( 4 < f'(z_1) < 99 < f'(z_0) \).

It is clear that \( f \) satisfies assumptions (A.1)-(A.5) of the previous section. Moreover, it is known that \( \lambda_{02} = x_{02}^2 \approx 49 < f'(\infty) < 100 \approx x_{03}^2 = \lambda_{03} \). Therefore by Corollary 5.2 we obtain that \( \nu(f'(\infty)) \) is odd.

Taking into account assumption (7) and Lemma 5.5 we obtain that \( f'(z_0) > f'(z_1) > \lambda_0(\Omega_1) \) and \( f'(z_0) > f'(\infty) \). Now it is easy to verify that under the above assumptions \( f \) satisfies assumption (1) of Theorem 4.1.3. Thus there exists at least one nonconstant weak solution of equation (5.1).

If there exists exactly one \( z_0 \in Z \) such that \( f'(z_0) > \lambda_0(\Omega_1) \), then in order to use Theorem 4.1.3 we have to replace assumption (7) with the following assumption:

\[
(7') \text{ there exists } k', n' \in \mathbb{N} \text{ such that } f'(z_0) < \lambda_{k'n'} < 50 \ (99 < \lambda_{k'n'} < f'(z_0)).
\]
Indeed, with assumption (7) replaced by assumption (7') the assumption (2) of Theorem 4.1.3 is fulfilled.

Example 5.2. Consider equation
\[
\begin{align*}
-\Delta u &= f(u) \quad \text{in } \Omega_2, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega_2,
\end{align*}
\]
where \( f \) satisfies the following assumptions:
1. \( f \in C^1(\mathbb{R}, \mathbb{R}) \),
2. \( |f'(x)| \leq a + b|x|^4 \) for some \( a, b > 0 \),
By Corollary 5.1 we obtain \( \lambda \) and Lemma 5.5 we obtain \( \lambda_f \) for all \( z \in Z \).

Combining assumptions (5.b), (5.c) with Lemma 5.5 we obtain that

It is clear that \( f \) satisfies assumptions (A.1)-(A.5) of the previous section. Moreover, by assumption (7) and Lemma 5.5, \( f'(z_0) > \lambda_0(\Omega_2) \). Applying Theorem 4.1.2 we obtain nonconstant weak solutions of equation (5.2).

Define \( Z_+ = \{ z \in Z \ | \ f'(z) > 0 \} \) and assume that \( 0 < f'(z) < 9 \), for every \( z \in Z_+ \).

Since \( x_{01} \simeq 3.83, f'(z) < \lambda_{001} = x_{01}^2 \) and \( f'(z) < \lambda_{010} = \pi^2 \). By Lemmas 5.1, 5.2 we obtain \( \{ \lambda_{0mj} \in \sigma(\Delta, \Omega_2) : \lambda_{0mj} < 9 \} = \{ \lambda_{000} \} \). Therefore, by Corollary 5.2, we obtain that \( \nu(f'(z)) \) is odd for every \( z \in Z_+ \). Notice that assumptions of Theorem 4.1.1 are not fulfilled. In other words we can not apply the Leray-Schauder degree to obtain the existence of nonconstant weak solutions of equation (5.2).

**Example 5.3.** Consider equation (5.2) and assume that

1. \( f \in C^1(\mathbb{R}, \mathbb{R}) \),
2. \( | f'(x) | \leq a + b|x|^4 \) for some \( a, b > 0 \),
3. \( f(t) = f'(\infty)t + o(|t|) \), where \( |t| \to \infty \),
4. \( #Z < \infty \),
5. there exists \( z_0 \in Z \) and \( k' \in \mathbb{N} \) such that:
   - \( f'(z_0) \notin \sigma(\Delta, \Omega_2) \),
   - \( f'(z_0) > 2k'(k' + 1) \),
   - \( f'(z_0) < k'(k' + 2) \) for \( z \in (Z \cup \{\infty\}) \setminus \{z_0\} \).

It is clear that \( f \) satisfies assumptions (A.1), (A.2) and (A.3) of the previous section. Combining assumptions (5.b), (5.c) with Lemma 5.5 we obtain that \( f'(z) < \lambda_{k'01} < f'(z_0) \) for all \( z \in (Z \cup \{\infty\}) \setminus \{z_0\} \). Fix \( \lambda \in \sigma(\Delta, \Omega_2) \cap (0, \lambda_{k'01}) \). By Corollary 5.3, \( \nu(\Delta, \lambda) \simeq \mathbb{R}[1, k_1] \oplus \cdots \oplus \mathbb{R}[1, k_s] \) for some \( k_1, \ldots, k_s \in \mathbb{N} \cup \{0\} \).

We claim that \( k_i < k' \) for every \( 1 \leq i \leq s \). Suppose, contrary to our claim, that \( k_i \geq k' \) for some \( 1 \leq i_0 \leq s \). Then, by Corollary 5.3, there exist \( n_{i_0} \in \mathbb{N} \cup \{0\}, j_{i_0} \in \mathbb{N} \) such that \( \lambda_{n_{i_0}j_{i_0}} = \lambda \). From Lemmas 5.1, 5.2 we obtain

\[
\lambda_{n_{i_0}j_{i_0}} = (\pi n_{i_0})^2 + x_{k_10j_0}^2 + x_{k_01j_0}^2 \geq x_{k_10j_0}^2 \geq x_{k_01j_0}^2 = \lambda_{k_01}.
\]

By Corollary 5.1 we obtain \( \lambda = \lambda_{k_0n_{i_0}j_{i_0}} \geq \lambda_{k_001} \geq \lambda_{k'01} \), a contradiction. Thus \( k_i < k' \) for \( i = 1, \ldots, s \) and consequently \( \nu(\lambda)Z_{k'} = \emptyset \). Taking into account assumption (5.c) and Lemma 5.5 we obtain \( \nu(\nu(\Delta, f'(z)) = \emptyset \) for all \( z \in (Z \cup \{\infty\}) \setminus \{z_0\} \) such that \( f'(z) \in \sigma(\Delta, \Omega_2) \).

Notice that all the assumptions of Theorem 4.1.5 are satisfied. Applying this theorem we obtain the existence of at least one nonconstant weak solutions of equation (5.2).

Suppose now that (5.a) does not hold, i.e. \( f'(z_0) \in \sigma(\Delta, \Omega_2) \). In order to obtain the existence of weak nonconstant solutions of equation (5.2) we have to assume:

\[
(5.a') \nu(\Delta, f'(z_0))^{SO(2)} = \{0\} \text{ and } \nu(\Delta, f'(z_0))Z_{k'} = \emptyset.
\]
It is clear that under the above assumption and assumptions (1)-(3), (5.b) and (5.c) Theorem 4.1.5 holds. This assumption is equivalent to the following one

(5.a’’) \( f'(z_0) \neq \lambda_{0nj} \) for \( n, j \in \mathbb{N} \cup \{0\} \) and \( f'(z_0) \neq \lambda_{k''nj} \) where \( k'' = k'm \) for \( m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}, j \in \mathbb{N} \).

References


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